

CHEAP AND SINGULAR  $\bar{H}^2$  AND  $\bar{H}^\infty$  CONTROL PROBLEMS: A  
GENERALIZED EIGENPROBLEM APPROACH

by

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## **Dedication**

To my wife, Prisca, and my children, Kafi and Keenon  
with whom I have endured the most intense and demanding period of my life.

To my parents,  
who have given me a great deal of encouragement and support over the years

And to Almighty God  
who has “tested me sorely, but not unto death”.

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## Abstract

This dissertation focuses primarily on control system design issues for linear systems which have zeros on the  $j\omega$ -axis and at infinity. In particular, the LQ/LQG,  $H^\infty$  and inner/outer factorization problems are considered. The solution to these problems are well known when the pertinent model has no such zeros; however, numerous difficulties arise when this not the case.

In this dissertation, we approach these ‘singular’ problems by perturbing them into ‘regular’ problems (i.e. ones for which there are no  $j\omega$  axis zeros) by way of a scalar parameter  $\epsilon$ . We then determine the limiting solution to these regular problems as  $\epsilon \rightarrow 0$ . This is, of course, the so-called cheap control approach. Our analysis of these cheap control problems is significant in that it employs a simple modal framework and yet allows us to completely characterize the relevant solutions. The use of the descriptor representation for linear systems facilitates the description of improper solutions where necessary. For  $H^\infty$  problems we have shown how to exploit singularity to obtain proper solutions of reduced order.

We also discuss the design of what we call zero-canceling compensators. These compensators allow us to take a direct approach to solving singular control problems

by extracting the 'regular part' of these problems. By way of an example, we apply this approach to solving the inner-outer factorization problem for strictly proper transfer functions.

## Notation

|                    |  |
|--------------------|--|
| $\Re$              | Field of real numbers  |
| $\mathcal{C}$      | Field of complex numbers   |
| $\mathcal{C}^{0e}$ | the extended $j\omega$ -axis, $\{j\omega : \omega \in \Re\} \cup \{\infty\}$           |
| $H^2, H^\infty$    | Hardy spaces   |
| $\sigma_{\max}(A)$ | maximum singular value of the matrix $A$   |
| $\ \cdot\ _2$      | norm on $H^2$  |
| $\ \cdot\ _\infty$ | norm on $H^\infty$   |
| $A^T$              | Matrix transpose of $A$  |
| $G^*(s)$           | para-Hermitian transpose of the transfer function $G(s)$<br>i. e. $G^*(s) = G^T(-s)$ . |
| Im                 | Image  |
| Ker                | Kernel   |
| $I_n$              | the $n \times n$ identity matrix   |
| $0_{m \times r}$   | the $m \times r$ zero matrix   |

In addition, we say that  $G(s) \stackrel{s}{=} \left[ \begin{array}{c|c} -sE + A & B \\ \hline C & D \end{array} \right]$  iff the transfer function  $G(s)$  has the descriptor realization

$$E\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

# Chapter 1

## Introduction

### 1.1 Singular Control Problems: Definition and Motivation for the Study

This dissertation analyzes finite-dimensional linear time-invariant (FDLTI) systems with  $C^{0e}$  zeros and the difficulties often faced in the synthesis of controllers for these systems using  $H^2$  (LQ/LQG) and  $H^\infty$  techniques. Particular attention is paid to the significance of the infinite zero structure of FDLTI systems.

The analysis was partly motivated by a study of problems which arise in the application of  $H^\infty$  theory to the design of controllers for FDLTI systems. One such application of this theory is illustrated in Fig. 1.1 for the mixed sensitivity problem [6]. Given a plant  $G(s)$  our task is to determine a controller  $K(s)$  which not only stabilizes  $G(s)$  but also makes  $T_{y_1 u_1}(s)$ , the transfer function from  $u_1$  to  $y_1$ , satisfy

$$\|T_{y_1 u_1}\|_\infty < 1 \quad (1.1)$$

The transfer function blocks  $W_1(s)$ ,  $W_2(s)$  in Fig 1.1 represent frequency dependent design weights on the closed loop sensitivity and complementary functions [6].

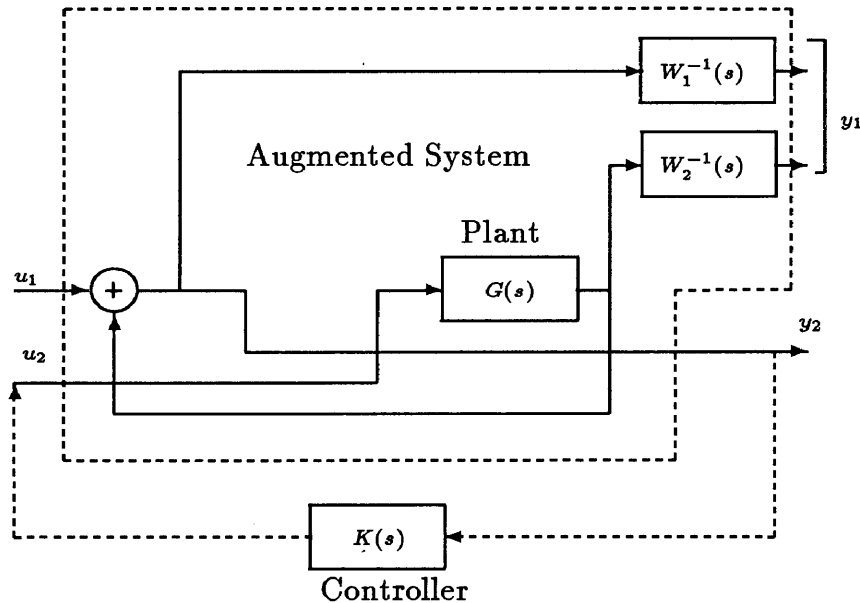


Figure 1.1: Setup for the Mixed Sensitivity Problem

The now widely accepted solution procedure discussed in Doyle *et al.*[14], for example, requires that the open loop transfer functions from  $u_1$  to  $y_2$  and  $u_2$  to  $y_1$  have no  $\mathcal{C}^{0e}$  zeros (this restriction also holds for  $H^2$  problems). If this restriction is violated, the designer must make suitable modifications so as to satisfy these requirements. In [6], for example, an  $H^\infty$  controller is synthesized for the HIMAT remotely piloted test aircraft by modifying the originally specified design weights  $W_1(s)$ ,  $W_2(s)$  in order to accommodate these restrictions.

The underlying issue in these restrictions is the insolubility of the underlying Riccati equations when the pertinent plant has  $\mathcal{C}^{0e}$  zeros; when this is the case, at least one of these Riccati equations either becomes ill-posed (if the  $\mathcal{C}^{0e}$  zeros are at

infinity) or does not have any stabilizing solution. This problem of the insolubility of the Riccati equation pervades other areas of control system analysis and design including problems such as the inner-outer and spectral factorization of a rational matrix.

To facilitate our discussion we introduce the following notation:

**Definition 1.1** *A problem will be called **singular** if the underlying Riccati equation cannot be solved because of the existence of  $C^{0e}$  zeros in the pertinent plant. Otherwise, the problem will be said to be **regular**.*

One of the main contributions of this work is that it effectively demonstrates how the difficulty with singular problems can be overcome by replacing the Riccati equation by a suitable generalized eigenvalue problem. The role of the Riccati equation solution matrix is then assumed by an appropriate set of eigenvectors. The reader should note that this approach has been known for some time (see Bucy *et al.*[5], for example). However, to the author's knowledge, this is the first time that the method has been so comprehensively applied to the problems discussed here. It is also found that many singular problems have solutions which are improper; this is handled by employing the descriptor (generalized state-space) representation for linear systems [64, 34, 3].



## 1.2 Overview and Background Literature

Singular LQ state feedback problems are often handled by perturbing them into ‘nearby’ regular ones. An example of this is the embedding

$$G(\epsilon, s) = \begin{bmatrix} G(s) \\ \epsilon I \end{bmatrix} \quad (1.2)$$

For  $\epsilon > 0$ ,  $G(\epsilon, s)$  has no zeros and the pertinent LQ problem is therefore regular.

Clearly, any solution obtained for some  $\epsilon > 0$  will be a suboptimal solution to the original problem (i. e. for  $\epsilon = 0$ ); however, a primary motivation for the embedding (1.2) is the hope that a characterization of the solution to the singular problem may be obtained by examining the behavior of the solution to the LQ for problem for  $G(\epsilon, s)$  as  $\epsilon \rightarrow 0$ . This new problem is often referred to in the literature as a **cheap** LQ state feedback control problem (see [30], for example). Note that this general approach of perturbing the plant by way of a scalar parameter  $\epsilon$  can also be applied to other singular problems; in such cases the terminology ‘cheap’ will also be used in reference to the pertinent new problem.

Cheap and singular LQ state feedback problems have been studied extensively over the last two decades. However, most of the analysis has focused on the case where the only  $\mathcal{C}^{0e}$  zeros are at infinity. The reason for this is that when there are finite imaginary axis zeros, analysis of the asymptotic root loci [30, 28, 53, 66] suggests that pole/zero cancellation of these  $\mathcal{C}^{0e}$  zeros must occur. This is, in general, undesirable since the cancellation takes place on the  $j\omega$ -axis. Moreover, for  $\epsilon$  small enough those closed loop poles which approach the troublesome zeros are lightly

damped; this is also undesirable. On the other hand, when the only zeros are at infinity, the asymptotic properties are generally much more attractive since the root loci form Butterworth patterns of various orders and magnitudes as  $\epsilon \rightarrow 0$  [30] in the right half plane (RHP). Despite the limitation of having finite  $C^{0e}$  zeros, the analysis in this work is believed to be useful as a starting point for the characterization of all suboptimal solutions to the relevant  $H^2$  and  $H^\infty$  problems. In view of what has just been said, however, it should be noted that the ensuing references to the literature on cheap and singular problems pertain specifically to the case where all  $C^{0e}$  zeros are infinite, except where otherwise stated.

The more successful attempts at solving cheap and singular LQ state feedback problems have employed either singular perturbation techniques [44, 51] or geometric control theory [15, 23, 59]; a neat characterization of the problem and its solution has been phrased in terms of the dissipation inequality [5, 24, 52]. All of these methods have identified the role of the infinite frequency structure of FDLTI systems in the cheap control problem. The analysis carried out in this dissertation displays this reliance on the infinite frequency structure in a very clear fashion through the determination of the eigenstructure of a particular matrix pencil; this also results in easy, reliable computation of the solution.

By far the most conclusive results on the singular LQG problem have arisen from the spectral factorization approaches of [29] and [55], for example. Frequency domain techniques are employed in these works and attention is restricted to the

case of infinite  $C^{0e}$  zeros. By contrast, our results rely upon a simple generalization of the commonly used procedures based on the solution to two Riccati equations and directly yields a realization of the optimal controller in descriptor form. Our analysis allows for all  $C^{0e}$  zeros. Apart from this, the results indicate that in those situations where the the optimal compensator is of reduced order,  $\delta < n$ , (see [41], for example) the solution automatically produces a controller of order  $\delta$ . This is in the vein of the result obtained by Friedland [17] by analyzing what is essentially a cheap LQG problem.

The singular  $H^\infty$  problem has been considered elsewhere (see [67, 43, 47, 42], for example). Zhou *et al.*[67] and Petersen *et al.*[43], for example treat the singular  $H^\infty$  problem with state feedback by perturbing the plant as described above for the cheap LQ state-feedback problem. It should be noted that for regular  $H^\infty$  problems, it is often necessary to iterate on the specified  $H^\infty$  bound,  $\gamma$ , in order to determine the optimal solution. It is therefore apparent that one drawback with the cheap control approach to the singular  $H^\infty$  problem is that now, one must perform a double iteration on both  $\gamma$  and  $\epsilon$ . O'Young *et al.* [42] seemingly consider a different perturbation of the problem by adding a constant control weighting which specifies a bound on the controller gain; although the authors do not specifically state it, this method also requires an iteration involving this specified bound, since there is no guarantee that a solution exists for their preset bound of  $\epsilon = 1$ . Despite this, the

approach will form the nucleus of a part of our analysis; in addition we will solve the attendant cheap  $H^\infty$  problem.

Other approaches to solving the singular  $H^\infty$  problem exist. In [47], for example, Safonov proposes two methods. The first entails the use of a bilinear transformation which effects a shift in the  $j\omega$ -axis. The transformed plant then has no zeros at infinity, or, if the transformation is carefully chosen, on the  $j\omega$ -axis for that matter. The solution may then be obtained by employing the established methods in [14, 19] and the required controller subsequently recovered by back-transformation. Alternatively, the author suggests high frequency modifications of the design weights to make the augmented system non-strictly proper.

More recently, a few authors have reported significant progress by employing new approaches to the problem. For example, Hara et al. [22] have reformulated the solution to the one-block case in a descriptor setting and have removed the restriction of  $j\omega$ -axis zeros. Kimura et al. [27] have employed a generalization of the theory of J-lossless systems in  $H^\infty$  synthesis and reported some preliminary results in the situation where the zeros at infinity are simple, i. e. of order at most unity.

Stoorvogel [57] has reported success in the singular case by use of a transformation which isolates those subsystems (strongly controllable subsystems) of  $G(s)$  which make problem singular; the problem is then transformed into an almost disturbance decoupling for which a solution is known [56, Ref 16]. The significance of

the work in [57] is that it directly exploits, for the first time, the underlying structural issues pertaining to the singular  $H^\infty$  problem, whereas previously published efforts have essentially dealt with a nearby plant for which the problem is regular. In [57, 56] the problem of the solvability of the Riccati equation is circumvented by employing an “inverse-free” representation of these equations which takes the form of quadratic matrix inequalities; these are related to the linear matrix inequalities which have proven useful in singular LQ problems [5, 52, 65].

Another possible approach to solving these singular problems involves the use of cascade compensators to cancel the offending zeros. This renders the problem regular and therefore amenable to solution by standard methods. The solution to the original problem can then be found by incorporating the compensators into the solution to the regular problem. This is demonstrated for the  $H^\infty$  problem in Fig. 1.2; if a suitable controller  $\hat{K}(s)$  can be determined for the regular problem  $\hat{G}(s)$ , then the controller  $U\hat{K}Y(s)$  is a solution to the original problem  $G(s)$ . While there are many details which remain to be worked out, the first step involves the determination of these zero canceling compensators.

For the most part, we will be concerned with the problem of infinite  $C^{0e}$  zeros. While there may be practical concerns pertaining to the cancellation of finite  $C^{0e}$  zeros, we feel that, at the very least, the analysis is important from a theoretical point of view. In addition, we note that for feedback design problems, such as  $H^2$  or  $H^\infty$  design problems, it can be shown that if a (possibly suboptimal) controller  $K(s)$

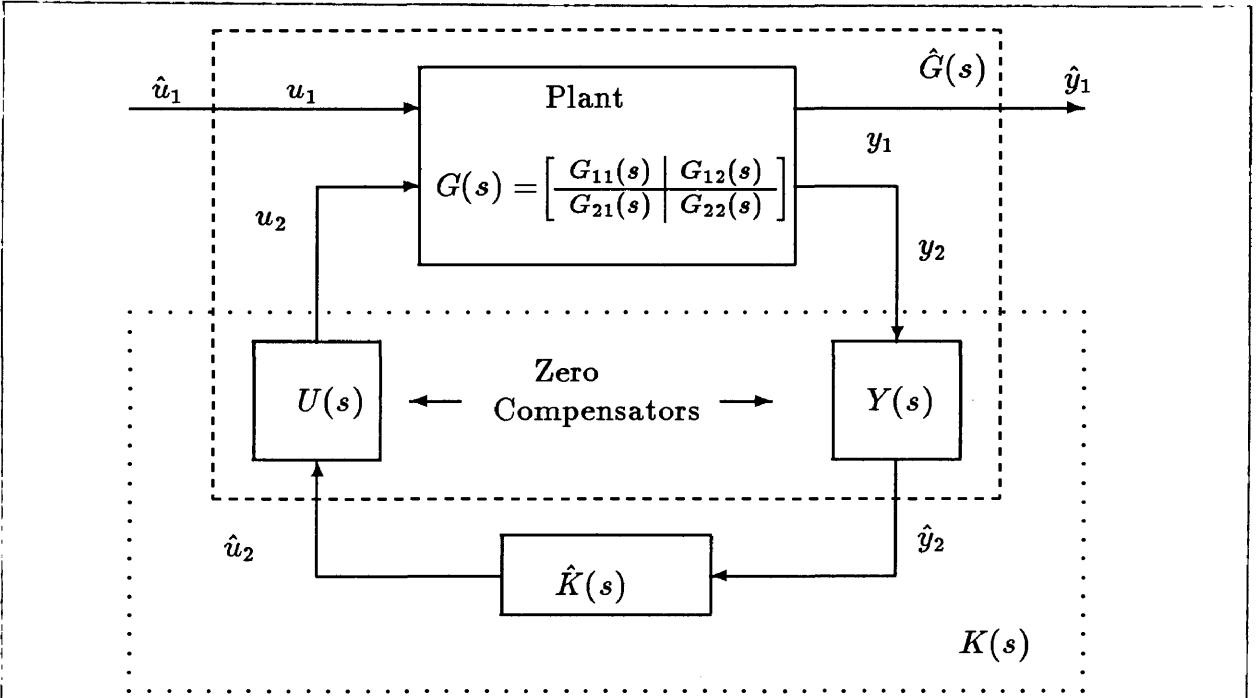


Figure 1.2: Proposed Solution to the Singular  $H^\infty$  Problem using Zero Compensation

exists without the embedded cancellation structure, then a corresponding  $\hat{K}(s)$  also exists. This will be discussed at greater length in a future work; for now, we focus our attention on the design of the required compensators.

We begin by considering the problem of obtaining a compensator  $U_\infty(s)$  for the proper  $m \times r$  transfer function  $G(s)$  such that  $G_c(s) = G(s)U_\infty(s)$  has no zeros at infinity; we shall call such a compensator an infinite zero compensator (IZC) of  $G(s)$ . We will be interested in those compensators for which  $G_c(s)$  has the same poles as  $G(s)$ ; as such, the procedure which we shall describe will result in  $U_\infty(s)$  being polynomial. We subsequently show how the theory developed can be easily extended to the case where  $G(s)$  has zeros on  $\mathcal{C}^0 := \{j\omega, \omega \in \mathfrak{R}\}$ . A

compensator,  $U_0(s)$  which corrects for the zeros of  $G(s)$  on  $\mathcal{C}^0$  will be termed a  $\mathcal{C}^0$  zero compensator ( $\mathcal{C}^0\text{ZC}$ ). Clearly, we can compensate for zeros on  $\mathcal{C}^{0e}$  by combining both procedures; in keeping with the terminology already introduced, we shall call the pertinent compensator a  $\mathcal{C}^{0e}$  zero compensator ( $\mathcal{C}^{0e}\text{ZC}$ ).

It should be noted that the solution to the IZC problem described above is by no means unique. Moreover, simple procedures for constructing  $U_\infty(s)$  have been known for quite a long time. For example, the cascade of differentiators generated by the Silverman inversion algorithm [54] represents, perhaps, the most familiar solution to the problem. The difficulty with this method, however, is that it requires recursive operations on the Markov parameters of  $G(s)$  and, as such, does not generally lend itself to computations which are numerically stable. In our analysis, we utilize the relationship between the infinite zero structure of  $G(s)$  [63] and the infinite eigenstructure of the associated Rosenbrock system matrix to develop a direct, numerically reliable method for the computation of  $U_\infty(s)$ .

The IZC approach is essentially the one taken by Safonov in [47] for adjusting  $H^\infty$  design weights which lead to ill-posed design problems. Here we improve upon the technique proposed in [47] by giving a state-space realization of the required compensator. Moreover, multiple zeros are handled in a single step as opposed to iteratively as in [47].

Another work worthy of mention is that of Van Dooren [13] where the author describes “numerically controlled” procedures for solving general factorization problems by using pole/zero cancellations. However, the work does not directly treat the situation when the zeros or poles are at infinity but, the author does indicate that in this case “the solution is more involved but can be constructed also”. Although our basic approach is different to that of Van Dooren, it leads to a procedure which is, for all intents and purposes, an extension of Algorithm 4.1 of [13] to the case of infinite zeros. As such, our efforts in this area may be considered to be somewhat complementary to [13].

By way of example, we apply the zero cancellation approach to develop a procedure for obtaining an inner-outer factorization (IOF) of a proper transfer function  $G(s)$  which is allowed to have zeros on  $\mathcal{C}^{0e}$ .

Well-known procedures exist for the IOF problem in the case where  $G(s)$  has no  $\mathcal{C}^{0e}$  zeros (see Chiang *et al.* [6] or Francis [16], for example). In our analysis, we lift the restriction on the zero structure of  $G(s)$  by employing a  $\mathcal{C}^{0e}\text{ZC}$ ,  $U_{0e}(s)$ , to cancel the offending zeros. The orthodox IOF procedures can now be applied and the problem solved by absorbing the function  $U_{0e}^{-1}(s)$  in the outer factor (of course, we must ensure that  $U_{0e}(s)$  has all its zeros in the LHP). The only other similar work of which we are aware is that of Hara *et al.* [22] who treat the problem in a descriptor framework. However, their procedure can only be applied to square transfer functions; non-square transfer functions require preliminary “squaring up”



by augmentation with suitable rows or columns. The method described here, on the other hand, may be directly applied to any transfer function regardless of dimension and can also be easily applied to other problems (e. g. the GCD extraction and squaring down problems in Le and Safonov [33]) where restrictions on the transfer function zeros are treated by employing either a preliminary bilinear map or the differentiator cascade generated by the Silverman algorithm.

### 1.3 Chapter Summary

In Chapter 2 we discuss the zero structure of FDLTI systems as described by the zero structure of the pertinent transfer function and system matrix. This theory is well covered in the literature (see Verghese *et al.*[63, 61] and MacFarlane [37], for example) for finite frequencies; however, it seems that the infinite frequency structure of FDLTI systems is not as well understood. The reader is referred to the works of Verghese *et al.*[63, 61] and Karcanias *et al.*[26] on this topic. The discussion in Chapter 2 is biased to a treatment of the infinite zeros structure of FDLTI systems and extends some of the concepts in these works.

In Chapter 3 we state and solve a general form of the cheap LQ state-feedback problem. The analysis given there forms the basis for the results developed in subsequent chapters pertaining to the cheap LQG and  $H^\infty$  problems. Since the limiting feedback control may not exist the solution to the cheap LQ problem is specified in terms of a modal subspace of a particular matrix pencil; this subspace is

shown to contain the limiting optimal state, costate and control input trajectories. The limiting cost matrix is also determined.

The cheap LQG and cheap  $H^\infty$  problems are dealt with in Chapters 4 and 5 respectively. We state formulae for the limiting system matrices of the relevant controllers. The descriptor framework [3, 12, 64] is employed to facilitate the representation of improper solutions. The limiting transfer functions are also specified and, in the case of the LQG problem, a formula for the limiting optimal cost is given.

In Chapter 6 we discuss the idea of obtaining solutions to singular problems by employing zero-canceling compensators. As discussed above the structure of these compensators are such that they cancel particular zeros of a given transfer function,  $G(s)$ ; this includes zeros at infinity. The application of the theory to  $H^2$  and  $H^\infty$  problems is not fully treated in this work; however, an illustrative example is given.

The theory is applied to the singular inner/outer factorization problem in Chapter 7.

## 1.4 Main Contributions

The main contributions in this dissertation are as follows:

(i) The power of the generalized eigenproblem approach as an alternative to the usual Riccati equation approach is demonstrated. This allows us to handle singular problems with the same ease as regular problems are handled. It is also expected

that there would be increased numerical stability particularly in problems which are “nearly singular”.

(ii) The generalization of the perturbation approach to solving singular  $H^\infty$  problem, as posited in [43, 67], for example, to include the case of measurement feedback. This is a modest improvement on the work by O’Young *et al.*[42].

(iii) It is shown that the solutions to certain cheap  $H^\infty$  problems may be of reduced order and proper. The result is easily derived when the only relevant  $C^0$  zeros are at infinity and are of maximum order 1. An explicit formula for one special case is derived and we also outline a simple strategy which allows one to obtain a reduced order ( $< n$ )  $H^\infty$  controller when these zeros are of higher order.

(iv) The limiting form of cheap LQG compensators is also derived. Explicit formulae for the limiting transfer function and associated cost are given.

(v) A numerically reliable method is given for determining an inner-outer factorization of transfer functions with  $j\omega$ -axis or infinite zeros.

(vi) The concept of the eigenvector structure associated with the infinite zeros of a transfer function  $G(s)$ , as first defined by Verghese *et al.*[63], is significantly developed and clarified. Necessary and sufficient conditions for a set of vectors to be a set of eigenvectors of  $G(s)$  at infinity are specified.

## Chapter 2

# Theoretical Review: Infinite Zeros of FDLTI Systems

### 2.1 Introduction

In this chapter we discuss the concept of the infinite zeros of FDLTI systems, a topic which will be of great significance to the analyses carried out in later chapters. We begin with an analysis of the infinite zero structure of transfer function matrices; in doing so we make a few clarifications and extensions to the classic works of Verghese *et al.*[63, 61] where the finite and infinite zero eigenvectors of a transfer function are first defined. Subsequently, we discuss the relationship between the eigenstructure of a transfer function  $G(s) = C(sI - A)^{-1}B + D$  and that of its Rosenbrock system matrix

$$M_G(s) = \left[ \begin{array}{c|c} -sI + A & B \\ \hline C & D \end{array} \right] \quad (2.1)$$

This requires a preliminary overview of the issue of the finite and infinite eigenstructure of matrix pencils. This particular topic is summarized in this chapter for

the sake of completeness, but is discussed at length in the works of Lewis [34], Van Dooren [12, 11] and Verghese [64].

## 2.2 Zeros of Transfer Function Matrices

Typically the pole/zero structure of any transfer function is obtained from its Smith-McMillan form [25, 37]. However the unimodular matrices employed to effect this transformation invariably destroy information concerning the infinite frequency structure of the system [25]. This situation is typically resolved by determining instead the zero structure at  $\rho = 0$  of  $G(\frac{1}{\rho})$ . In order to circumvent this special treatment of zeros at infinity, Verghese et al. [63, 61] have proposed a slightly different definition of pole/zero structure at  $s = z, z \in \mathcal{C} \cup \{\infty\}$  based on the nonunique decomposition

$$G(s) = L^{-1}(s)D(s)R^{-1}(s) \quad (2.2)$$

where  $D(s)$  is a rational matrix of the same dimension as  $G(s)$  of the form

$$D(s) = \begin{bmatrix} \text{diag}\{d_1(s), d_2(s), \dots, d_r(s)\} \\ 0_{(m-r) \times r} \end{bmatrix} \quad (2.3)$$

The matrices  $L(s)$  and  $R(s)$  are nonsingular rational matrices with no poles or zeros at  $s = z$ , where  $z$  is a zero or pole of  $G(s)$ . In the case where  $z = \infty$ ,  $L(s)$  and  $R(s)$  are **biproper**, i. e. they are proper and have proper inverses. This decomposition is always possible,  $L(s)$  and  $R(s)$  being non-unique. However, as we will shortly

discuss, the decomposition (2.2) preserves the zero structure of  $G(s)$  at infinity, i. e.  $D(s)$  has the same infinite zero structure of  $G(s)$ .

Now suppose that  $R(s)$  and  $L(s)$  in (2.2) is biproper and let  $k_i$  be the relative degree of  $d_i(s)$  with  $k_1 \geq k_2 \geq \dots \geq k_r \geq 0$ . Then given  $G(s)$  these  $k_i$ ,  $i \in \underline{r}$  are unique and determine the zero structure of  $G(s)$  at infinity. This fact has been established by Verghese [62] by demonstrating the equivalence of this definition of zero structure at infinity to that obtained from the Smith-McMillan form of  $G(\frac{1}{s})$ . Note also that the definition of infinite zero structure by way of biproper matrices, as in (2.2), is now generally accepted [40].

In what follows, we shall say that  $k_i$  is the order of the  $i$ -th infinite zero of  $G(s)$ . Note that if  $k_i = 0$ , the corresponding diagonal element  $d_i(s)$  is biproper and therefore has no infinite zeros (i. e. all of its zeros are finite). Thus  $G(s)$  has no zeros at infinity iff  $k_i = 0 \forall i \in \underline{r}$ .

We also need to introduce the concept of eigenvector chains at infinity for a given transfer function  $G(s)$ . To this end, first note that since  $R(s)$  has a Laurent expansion:

$$R(s) = \sum_{j=0}^{\infty} \frac{R_j}{s^j} \quad (2.4)$$

where  $R_0$  is nonsingular since  $R(s)$  is, by definition, biproper.

**Definition 2.1** Let  $G(s)$  have an infinite zero of order  $k_i$  and let the vector set  $\{f_{i_0}, \dots, f_{i_{k_i}}\}$  be such that  $\exists$  some biproper  $R(s), L(s)$  which effect the decomposition (2.2) and for which the  $i$ -th column of  $R(s)$  is given by

$$R_i(s) = f_{i_0} + f_{i_1} s^{-1} + \dots + f_{i_{k_i}} s^{-k_i} + \sum_{j=k_i+1}^{\infty} R_{i_j} s^{-j} \quad (2.5)$$

Then

(i)  $f_{i_0}$  will be said to be an eigenvector corresponding to the  $i$ -th zero of  $G(s)$  at infinity. In addition, the vector  $f_{i_j}, j \in \underline{k_i} \cup \{0\}$  will be said to be a grade  $j + 1$  eigenvector corresponding to the  $i$ -th zero of  $G(s)$  at infinity.

(ii) the vector set  $\{f_{i_0}, \dots, f_{i_{k_i}}\}$  will be said to be an eigenvector chain corresponding to the  $i$ -th zero of  $G(s)$  at infinity.

*Comment:* This definition of eigenvector chains at infinity is a modification of a similar definition due to Verghese [63]; specifically, the definition in [63] coincides with that of Definition 2.1 but excludes the last element  $f_{i_{k_i}}$ . Definition 2.1 simplifies our analysis by allowing us to avoid any special treatment of infinite zeros of order  $k_i = 0$ . Moreover, as we shall soon see, it allows us to firmly establish the link between the infinite frequency zero structure of  $G(s)$  and that of its state-space system matrix. The definition of eigenvector grade is made to correspond with a similar definition for matrices [32] and matrix pencils (see Section 2.3).

Verghese et al. [63] have also shown that any eigenvector chain at infinity is related to the Markov parameters [25] of  $G(s)$  as follows (see also Nijmeijer et al. [40]):

$$\begin{bmatrix}
D & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\
CB & D & 0 & \dots & \dots & \dots & \dots & 0 \\
CAB & CB & D & 0 & \dots & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & D & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & D & 0 & 0 \\
CA^{k_i-2}B & CA^{k_i-3}B & \dots & \dots & \dots & CB & D & 0 \\
CA^{k_i-1}B & CA^{k_i-2}B & \dots & \dots & \dots & \dots & CB & D
\end{bmatrix}
\begin{bmatrix}
f_{i_0} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
f_{i_{k_i-1}} \\
f_{i_{k_i}}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
0 \\
M_i
\end{bmatrix}, \quad M_i \neq 0 \tag{2.6}$$

The following result is important in that it shows that (2.6) is included in a set of sufficient conditions which a collection of vectors  $\{f_{i_0}, \dots, f_{i_{k_i}}\}$  must satisfy in order to be an eigenvector chain at infinity for a given transfer function  $G(s)$ .

**Lemma 2.1** *Suppose that  $G(s)$  has infinite zeros of respective orders  $k_i$ ,  $i \in \underline{r}$ . Suppose also that, for each  $i \in \underline{r}$ , there exist vectors  $f_{i_0}, \dots, f_{i_{k_i}}$  which satisfy (2.6) for which*

$$\text{rank} \begin{bmatrix} f_{1_0} & \dots & f_{r_0} \end{bmatrix} = r. \tag{2.7}$$

and

$$\text{rank } M = r \tag{2.8}$$



where the  $i$ -th column of  $M$  is  $M_i$  as given by the last row of Eq (2.6), i. e.

$$M_i = \begin{bmatrix} CA^{k_i-1}B & \dots & CB & D \end{bmatrix} \begin{pmatrix} f_{i_0} \\ \vdots \\ \vdots \\ f_{i_{k_i}} \end{pmatrix} \quad (2.9)$$

Then the vector sets  $\{f_{i_0}, \dots, f_{i_{k_i}}\}$ ,  $i \in \underline{r}$ , form eigenvector chains for  $G(s)$  at infinity.

*Proof:* Define

$$f_i(s) := f_{i_0} + f_{i_1}s^{-1} + \dots + f_{i_{k_i}}s^{-k_i} \quad (2.10)$$

and

$$F(s) := \begin{bmatrix} f_1(s) & \dots & f_r(s) \end{bmatrix} \quad (2.11)$$

Also let

$$D_c(s) := \text{diag}\{s^{k_1}, \dots, s^{k_r}\} \quad (2.12)$$

Then we claim that  $G_F(s) := GF D_c(s)$  is proper and the condition (2.8) implies that  $G_F(s)$  has no infinite zeros; i. e. if

$$G_F(s) = D_F + C_F(sI - A_F)^{-1}B_F \quad (2.13)$$

then  $D_F$  is injective. To see this note that the  $i$ -th column of  $G_F(s)$  is

$$G_{F_i}(s) = \left[ D + \frac{1}{s}CB + \frac{1}{s^2}CAB + \dots \right] \begin{bmatrix} f_{i_0}s^{k_i} + f_{i_1}s^{k_i-1} + \dots + f_{i_{k_i}} \end{bmatrix} \quad (2.14)$$

To show that  $G_F(s)$  is proper, simply look at the polynomial part of  $G_{F_i}(s)$  (we let  $S := sI_r$ ):

$$\begin{aligned}
\text{pol}(G_{F_i}(s)) &= \\
&= \text{pol} \left( \begin{bmatrix} D & CB & \dots & CA^{k_i-2}B & CA^{k_i-1}B \end{bmatrix} \begin{bmatrix} I \\ S^{-1} \\ \vdots \\ S^{-k_i+1} \\ S^{-k_i} \end{bmatrix} \begin{bmatrix} S^{k_i} & S^{k_i-1} & \dots & S & I \end{bmatrix} \begin{bmatrix} f_{i0} \\ f_{i1} \\ \vdots \\ f_{ik_i-1} \\ f_{ik_i} \end{bmatrix} \right) \\
&= \text{pol} \left( \begin{bmatrix} D & CB & \dots & CA^{k_i-2}B & CA^{k_i-1}B \end{bmatrix} \begin{bmatrix} S^{k_i} & S^{k_i-1} & \dots & S & I \\ S^{k_i-1} & & \dots & I & S^{-1} \\ \vdots & \dots & \dots & \dots & \vdots \\ S & I & \dots & \dots & S^{-k_i+1} \\ I & S^{-1} & \dots & S^{-k_i+1} & S^{-k_i} \end{bmatrix} \begin{bmatrix} f_{i0} \\ f_{i1} \\ \vdots \\ f_{ik_i-1} \\ f_{ik_i} \end{bmatrix} \right) \\
&= \begin{bmatrix} D & CB & \dots & CA^{k_i-2}B & CA^{k_i-1}B \end{bmatrix} \begin{bmatrix} S^{k_i} & S^{k_i-1} & \dots & S & I \\ S^{k_i-1} & & \dots & I & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ S & I & \dots & \dots & 0 \\ I & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{i0} \\ f_{i1} \\ \vdots \\ f_{ik_i-1} \\ f_{ik_i} \end{bmatrix} \\
&= \begin{bmatrix} S^{k_i} & S^{k_i-1} & \dots & I \end{bmatrix} \begin{bmatrix} D & 0 & 0 & \dots & \dots & \dots & 0 \\ CB & D & 0 & \dots & \dots & \dots & 0 \\ CAB & CB & D & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & D & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & D & 0 \\ CA^{k_i-2}B & CA^{k_i-3}B & \dots & \dots & \dots & CB & D \end{bmatrix} \begin{bmatrix} f_{i0} \\ f_{i1} \\ \vdots \\ f_{ik_i} \end{bmatrix} \\
&= M_i \tag{2.15}
\end{aligned}$$

Hence  $G_F(s)$  is proper and  $D_F = M$  which is injective by assumption. It therefore follows that  $G_F(s)$  has no infinite zeros. In addition, we note that by (2.7)  $F(s)$  is biproper.

Now let  $\hat{D}_F$  be such that the matrix  $\begin{bmatrix} D_F & \hat{D}_F \end{bmatrix}$  is nonsingular. Also, let

$$L(s) := \begin{bmatrix} G_F(s) & \hat{D}_F \end{bmatrix}. \tag{2.16}$$

Then  $L(s)$  is biproper by construction and

$$GF(s) = G_F(s)D_c^{-1}(s) \quad (2.17)$$

$$= L(s) \begin{bmatrix} I_r \\ 0 \end{bmatrix} D_c^{-1}(s) \quad (2.18)$$

$$= L(s) \begin{bmatrix} D_c^{-1}(s) \\ 0 \end{bmatrix} \quad (2.19)$$

Now note that since  $F(s)$  and  $L(s)$  are biproper by construction, comparison with (2.2) then gives the required result.  $\square$

*Comment:* The proof to Lemma 2.1 provides us with some insight into the significance of the condition (2.8). Specifically, the discussions there indicate that the columns of  $M$  are also the first  $r$  columns of some  $L^{-1}(s)$  at  $s = \infty$  (see (2.16)) for which the decomposition (2.2) results in

$$D(s) = \begin{bmatrix} \text{diag}\{s^{-k_1}, \dots, s^{-k_r}\} \\ 0 \end{bmatrix} \quad (2.20)$$

Hence, by the construction given in the proof to Lemma 2.1, (2.7) is necessary and sufficient to guarantee the existence of a biproper  $R(s)$  in (2.2), while (2.8) guarantees the existence of an  $L(s)$  which is also biproper.

Lemma 2.1 leads directly to the next result which states that (2.6, 2.7, 2.8) give a complete characterization of eigenvector chains at infinity.

**Lemma 2.2** *Suppose that  $G(s)$  has infinite zeros of respective orders  $k_i$ ,  $i \in \underline{r}$ . Then the relations (2.6, 2.7, 2.8) are necessary and sufficient conditions for any set*

of vector chains  $\{f_{i_0}, \dots, f_{i_k}\}$ ,  $i \in \underline{r}$  to be a complete set<sup>1</sup> of eigenvectors chains at infinity for  $G(s)$ .

We shall forego the proof to Lemma 2.2 until Chapter 6.

Finally, we would like to stress that these eigenvector chains are not unique, a fact which is easily ascertained by recalling that the decomposition (2.2) is itself non-unique. In particular, since  $R(s)$  can be diagonally scaled on the right while still preserving (2.2), it is clear that each eigenvector chain can be multiplied by an arbitrary scalar. This is, of course, in keeping with the expected properties of a set of eigenvectors. We refer the reader to Appendix A where we discuss the issue of eigenvector uniqueness in more detail.

## 2.3 Eigenstructure of Matrix Pencils

In this section we review the theory of infinite zeros of linear time-invariant systems. An important component in this discussion is the concept of eigenvalues and eigenvectors of matrix pencils. This topic is covered in detail elsewhere (see [18, 32, 34], for example) and so will only be briefly considered here.

First we define the finite eigenvalues of the regular pencil  $M(s) = -sE + A$  to be the zeros of the polynomial equation  $\det M(s) = 0$  if this polynomial is not a constant. Note that  $M(s)$  may have eigenvalues at infinity if  $E$  is singular [34]. We

---

<sup>1</sup>A set  $S_G$  of eigenvector chains at infinity will be said to be complete iff for each infinite zero of  $G(s)$   $\exists$  a corresponding eigenvector chain in  $S_G$

also observe that [34] there exist nonsingular matrices  $R, L$  which reduce  $M(s)$  to the Weierstrass canonical form defined by

$$LM(s)\Xi = \text{diag}\{sI - \Lambda_f, -s\Lambda_\infty + I\} \quad (2.21)$$

where  $\Lambda_f$  is a Jordan matrix whose eigenvalues correspond to the finite eigenvalues of  $M(s)$  and  $\Lambda_\infty$  is a nilpotent matrix in Jordan form. The pencil  $-s\Lambda_\infty + I$  thus contains the infinite frequency structure of  $M(s)$  and the number and sizes of the Jordan blocks in  $\Lambda_\infty$  determine the number and orders, respectively, of the infinite eigenvalues of  $M(s)$ . By definition the finite and infinite eigenspaces are the spaces spanned by the columns of  $\Xi_f$  and  $\Xi_\infty$  respectively where  $\Xi = \begin{bmatrix} \Xi_f & \Xi_\infty \end{bmatrix}$  is a partitioning of  $\Xi$  corresponding to the RHS of (2.21). Moreover,

$$A\Xi_f = -E\Xi_f\Lambda_f \quad (2.22)$$

$$A\Xi_\infty\Lambda_\infty = -E\Xi_\infty \quad (2.23)$$

Alternatively, we can rewrite these equations as follows:

$$(-sE + A)\Xi_f = E\Xi_f(-sI + \Lambda_f) \quad (2.24)$$

$$(-sE + A)\Xi_\infty\Lambda_\infty = E\Xi_\infty(-s\Lambda_\infty + I) \quad (2.25)$$

Suppose that  $\Lambda_{f_j}$  is a  $k_j \times k_j$  Jordan matrix of  $\Lambda_f$  such that  $\det(sI - \Lambda_{f_j}) = (s - z_j)^{k_j}$ ; also suppose that the corresponding columns of  $\Xi_f$  are respectively  $\begin{bmatrix} \xi_j^{(1)} & \dots & \xi_j^{(k_j)} \end{bmatrix}$ . Then  $z_j$  is said to be a finite zero of  $M(s)$  of order  $k_j$  and that  $\xi_j^{(k)}$ ,  $k \in \underline{k_j}$ , is a grade  $k$  eigenvector corresponding to the zero  $z_j$ .

Similarly, if  $\Lambda_{\infty_j}$  is a  $k_j \times k_j$  Jordan block of  $\Lambda_\infty$ , and the corresponding columns of  $\Xi_\infty$  are  $\xi_{\infty_j}^{(1)}, \dots, \xi_{\infty_j}^{(k_j)}$  then we say that  $z_j = \infty$  is a zero of  $M(s)$  of order  $k_j - 1$ .

Note that the order of each infinite zero is one less than the dimension of the corresponding Jordan block. We define  $\xi_{\infty_j}^{(k)}$  to be the grade  $k$  eigenvector at infinity of  $M(s)$  corresponding to the  $j$ -th infinite zero; moreover, the vectors  $\{\xi_{\infty_j}^{(i)}, i \in \underline{k_j}\}$  are said to form an eigenvector chain at infinity. From (2.23) these vectors also satisfy the relation

$$E\xi_{\infty_j}^{(1)} = 0, \quad E\xi_{\infty_j}^{(k)} = -A\xi_{\infty_j}^{(k-1)} \quad k = 2, \dots, k_j \quad (2.26)$$

We will need the following:

**Definition 2.2** *The left eigenvectors of the regular pencil  $M(s)$  are defined to be the eigenvectors of  $M^T(s)$ .*

When  $M(s)$  is a rectangular pencil of full column rank the eigenvalues and eigenvectors can be defined in a similar fashion. We note that there exist nonsingular constant matrices  $X, Y$  which transform such an  $M(s)$  into its Kronecker canonical form (KCF) [11, 18] i. e.

$$YM(s)X = \text{diag}\{sE_\eta + A_\eta, sE_f + A_f\} \quad (2.27)$$

where  $sE_\eta + A_\eta$  is injective on  $\mathcal{C} \cup \{\infty\}$  and the pencil  $sE_f + A_f$  is regular. The latter pencil clearly contains the column zero structure of  $M(s)$ . We therefore define the eigenvalues of  $M(s)$  to be the eigenvalues of  $sE_f + A_f$  (these include eigenvalues at infinity) which justifies our claim.

**Lemma 2.3** *The pencil  $M(s)$  has no infinite zeros iff all eigenvector chains at infinity are of maximum length 1.*

*Proof:* It is easily ascertained that the pencil  $M(s)$  has no zeros at infinity iff the regular part of its KCF (2.27) has Weierstrass form

$$-sE_f + A_f = M_1(s) = -sI + A_1 \quad \text{or} \quad -sE_f + A_f = M_2(s) = \text{diag}\{-sI + A_1, A_2\} \quad (2.28)$$

where  $A_2$  is nonsingular. In both cases, it is also easily verified that there are no infinite zeros since the order of the  $\det(-sE_f + A_f)$  is equal to  $\text{rank } E_f$ . The proof is completed by noting that  $M_1(s)$  has no eigenvector chains at infinity (or equivalently all chains at infinity are of zero length) while all eigenvector chains at infinity for  $M_2$  are of length 1.

□

The following definition introduces the concept of trivial infinite zeros:

**Definition 2.3** *Any infinite eigenvalue of  $M(s)$  of order  $k = 0$  will be said to be trivial. All other infinite eigenvalues will be called non-trivial.*

Before proceeding we remind the reader that the eigenvector chains are of significance to the solution of the equation

$$(-sE + A)\xi(s) = E\xi_0 \quad (2.29)$$

for any  $\xi_0 \in \mathfrak{R}^n$ . The following lemma summarizes the pertinent result for both the finite and infinite eigenspaces:

**Lemma 2.4** (Verghese *et al.* [63]) *For the equation (2.29) the following hold:*

(i) For  $\xi_0 = -\xi_j^{(l)}$   $l \in \underline{k}_j$ , corresponding to the finite eigenvalue  $\lambda_j$  of  $-sE + A$ , the solution  $\xi(s)$  is

$$\xi(s) = \sum_{i=1}^l \xi_j^{(l+1-i)} \frac{1}{(s - \lambda_j)^i}. \quad (2.30)$$

(ii) For  $\xi_0 = -\xi_{\infty j}^{(l)}$ ,  $l = 2, 3, \dots, k_j + 1$ , corresponding to an infinite eigenvalue of order  $k_j$ , the solution to (2.29) is given by

$$\xi(s) = \sum_{i=1}^{l-1} \xi_j^{(l-i)} s^{(i-1)}. \quad (2.31)$$

*Comment:* Let

$$\delta_{\infty} := \sum_{j=1}^r k_j \quad (2.32)$$

and note that for the infinite frequency spectrum, the solutions are polynomial and evolve in the  $\delta_{\infty}$ -dimensional subspace spanned by the the infinite eigenvectors of all grades but the highest. The vectors  $\xi_0$  lie in a  $\delta_{\infty}$ -dimensional subspace spanned by infinite frequency eigenvectors of all but the first grade. The distinct roles played by the two subspaces of the infinite frequency eigenspace of a pencil,  $M(s)$ , motivates the following

**Definition 2.4** Given any pencil  $M(s)$  we define the **lower infinite frequency eigensubspace** of  $M(s)$  to be the space spanned by all but the highest grade eigenvectors corresponding to all nontrivial infinite zeros of  $M(s)$ . Similarly, we define the **upper infinite frequency eigensubspace** of  $M(s)$  to be the space spanned by all but the first grade eigenvectors corresponding to all non-trivial infinite zeros of  $M(s)$ .



*Comment:* At this point we note that our discussion on the infinite zero structure of FDLTI systems has a geometric interpretation which is discussed in Commault *et al.*[7], for example. In particular, if the columns of  $\Xi^{(u)} = \begin{bmatrix} X^{(u)} \\ U^{(u)} \end{bmatrix}$  span the upper infinite frequency eigensubspace of  $M(s)$  with  $X^{(u)}$  being an  $n$ -rowed partition of  $\Xi^{(u)}$ , then it follows from the definition above that the  $\text{Im } X^{(u)} = \mathcal{V}_b^*/\mathcal{V}^*$  which is a geometric subspace defined in [7].

The next two lemmas will prove useful in the ensuing analysis.

**Lemma 2.5** *Let  $M(s)$  be a regular pencil with  $X$  being a matrix whose columns form a basis for the eigenspace corresponding to some eigenvalue  $\lambda_x \in \mathcal{C} \cup \{\infty\}$  of  $M(s)$ . Similarly, let  $Y$  be a matrix whose columns form a basis for the left eigenspace of  $M(s)$  corresponding to an eigenvalue  $\lambda_y \in \mathcal{C} \cup \{\infty\}$ .*

*Then if  $\lambda_x \neq \lambda_y \Rightarrow$*

$$Y^T(-sE + A)X \equiv 0. \quad (2.33)$$

*Proof:* Suppose first, that  $\lambda_x$  is finite. Then from (2.22, 2.23) there exist matrices  $\Lambda_y$  and  $\Lambda_x$  satisfying

$$Y^T A = \Lambda_y^T Y^T E \quad (2.34)$$

$$A X = E X \Lambda_x \quad (2.35)$$

where all the eigenvalues of  $\Lambda_y$  and  $\Lambda_x$  are at  $\lambda_y$  and  $\lambda_x$  respectively. Left multiply (2.35) by  $Y^T$  and substitute from (2.34) to get

$$Y^T EX\Lambda_x = Y^T AX = \Lambda_y^T Y^T EX \quad (2.36)$$

i. e.

$$(Y^T EX)\Lambda_x - \Lambda_y^T (Y^T EX) = 0 \quad (2.37)$$

Now note that by assumption, the eigenvalues of  $\Lambda_x$  and  $\Lambda_y$  do not correspond  $\Rightarrow$  by Lemma 1.5 of [31] that the unique solution to this last equation is  $Y^T EX = 0$ .

For  $\lambda_x = \infty$ , (2.35) becomes

$$AX\Lambda_x = EX \quad (2.38)$$

where  $\Lambda_x$  is now a nilpotent matrix. Again leftmultiply by  $Y^T$  and substitute from (2.34) to get

$$Y^T EX = \Lambda_y^T Y^T EX\Lambda_x \quad (2.39)$$

By repeated substitution from (2.34) and (2.38) we get

$$\left. \begin{aligned} Y^T EX &= \Lambda_y^T Y^T EX\Lambda_x \\ &= \Lambda_y^T Y^T AX\Lambda_x^2 \\ &= \Lambda_y^{2'} Y^T EX\Lambda_x^2 \\ &= \vdots \\ &= \Lambda_y^{n'} Y^T EX\Lambda_x^n \text{ \forall integers } n \end{aligned} \right\} \quad (2.40)$$

Since  $\Lambda_x$  is nilpotent then, by definition, there exists some integer  $n \geq 1$  such that  $\Lambda_x^n = 0$ . The result then follows. □

**Lemma 2.6** *Let*

$$\Xi_\infty = \left[ \begin{array}{c|c|c|c} \xi_1^{(1)} & \dots & \xi_1^{(k_1)} & \xi_2^{(1)} & \dots & \xi_2^{(k_2)} & \dots & \dots & \dots & \xi_r^{(1)} & \dots & \xi_r^{(k_r)} \end{array} \right] \quad (2.41)$$

be a matrix of infinite zero eigenvectors of the pencil  $M(s) = -sE + A$  corresponding to the nilpotent Jordan matrix  $\Lambda_\infty$  i. e.

$$M(s)\Xi_\infty\Lambda_\infty = E\Xi_\infty(-s\Lambda_\infty + I) \quad (2.42)$$

Define

$$\Xi_\infty^{(l)} := \begin{bmatrix} \Xi_{\infty_1}^{(l)} & \dots & \Xi_{\infty_r}^{(l)} \end{bmatrix} \quad (2.43)$$

and

$$\Xi_\infty^{(u)} := \begin{bmatrix} \Xi_{\infty_1}^{(u)} & \dots & \Xi_{\infty_r}^{(u)} \end{bmatrix} \quad (2.44)$$

where

$$\Xi_{\infty_j}^{(l)} := \begin{bmatrix} \xi_j^{(1)} & \dots & \xi_j^{(k_j-1)} \end{bmatrix} \quad (2.45)$$

and

$$\Xi_{\infty_j}^{(u)} := \begin{bmatrix} \xi_j^{(2)} & \dots & \xi_j^{(k_j)} \end{bmatrix} \quad (2.46)$$

Then  $\Xi_\infty^{(l)}$  and  $\Xi_\infty^{(u)}$  span the lower and upper infinite frequency eigensubspaces of  $M(s)$ , respectively, and

$$A\Xi^{(l)} = E\Xi^{(u)} \quad (2.47)$$

*Proof:* By definition  $\Xi_\infty^{(l)}$  and and upper infinite frequency eigensubspaces of  $M(s)$  respectively. To prove (2.47) it suffices to take the eigenvectors associated with a single Jordan block in (2.42) i. e.

$$M(s)\Xi_{\infty_j}\Lambda_{\infty_j} = E\Xi_{\infty_j}(-s\Lambda_{\infty_j} + I) \quad (2.48)$$

where  $\Lambda_{\infty_j}$  is the  $k_j \times k_j$  matrix given by

$$\Lambda_{\infty_j} = \begin{bmatrix} 0 & I_{k_j-1} \\ 0 & 0 \end{bmatrix} \quad (2.49)$$

and

$$\Xi_{\infty_j} = \left[ \xi_j^{(1)}, \dots, \xi_j^{(k_j)} \right] \quad (2.50)$$

Then

$$\left. \begin{aligned} \Xi_{\infty_j} &= \left[ \Xi_{\infty_j}^{(l)} \quad \xi^{(k_j)} \right] \\ \text{and} \\ \Xi_{\infty_j} &= \left[ \xi^{(1)} \quad \Xi_{\infty_j}^{(u)} \right] \end{aligned} \right\} \quad (2.51)$$

Substitution in 2.42) yields

$$A \left[ \Xi_{\infty_j}^{(l)} \quad \xi^{(k_j)} \right] \Lambda = E \left[ \xi^{(1)} \quad \Xi_{\infty_j}^{(u)} \right] \quad (2.52)$$

and the result follows by equating the last  $(k_j - 1)$  columns of (2.52).

□

We conclude the section with a brief discussion on pencils which are Rosenbrock system matrices in state space form i. e.

$$M(s) = -sE_M + A_M = \left[ \begin{array}{c|c} sI - A & -B \\ \hline C & D \end{array} \right]. \quad (2.53)$$

A concept which is of critical importance to our study is that of the invariant zeros of the system  $G(s) = C(sI - A)^{-1}B + D$ . The finite invariant zeros of this system are defined to be the finite eigenvalues of  $M(s)$  [37, 45]. For infinite invariant zeros, we employ the following definition:

**Definition 2.5 (Infinite Invariant Zeros)** *The infinite invariant zeros of  $\{A, B, C, D\}$  are defined to be the non-trivial infinite eigenvalues of  $M(s)$ .*

The definition is motivated by the fact that trivial infinite eigenvalues are associated with constant diagonal blocks in the KCF of  $M(s)$ . The implication for the system  $\{A, B, C, D\}$  is that even when  $D$  is injective (in which case there are no transmission zeros at infinity [61]), the pencil  $M(s)$  will have  $r$  trivial infinite eigenvalues. We also recall that for finite frequencies [37], the set of transmission zeros and invariant zeros coincide. The fact that this also holds in the infinite frequency case, with the definition posed, follows from Lemma 2.7 below. Observe that the lower and upper infinite frequency eigensubspaces of  $M(s)$  are spanned by all but the highest, respectively lowest, grade eigenvectors corresponding to the infinite invariant zeros of  $\{A, B, C, D\}$ .

**Lemma 2.7** *The following properties link the infinite zero structures of  $M(s)$  and  $G(s)$ :*

(i)  *$G(s)$  has  $l$  infinite zeros of order  $k_i$ ,  $i \in \underline{l} \Leftrightarrow M(s)$  has  $l$  infinite eigenvalues of respective orders  $k_i$ ,  $i \in \underline{l}$ .*

(ii) *The chain  $\{u_0, \dots, u_k\}$  is an eigenvector chain corresponding to a  $k$ -th order infinite zero of  $G(s)$ ,  $\Leftrightarrow$  the chain  $\{\xi^{(1)}, \dots, \xi^{(k+1)}\}$  is an eigenvector chain corresponding to a  $k$ -th order infinite eigenvalue of  $M(s)$ , where*

$$\left. \begin{aligned} \xi_i^{(1)} &= \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \\ \xi_i^{(q)} &= \begin{pmatrix} \sum_{j=1}^{q-1} A^{j-1} B u_{q-(1+j)} \\ u_{q-1} \end{pmatrix}, \quad q = 2, 3, \dots, k+1 \end{aligned} \right\} \quad (2.54)$$

*Proof:* See Appendix A. □

Finally, we have the following result

**Lemma 2.8** *The lower infinite frequency eigensubspace of the pencil  $M(s)$  in (2.53) lies in  $\text{Ker} \begin{bmatrix} C & D \end{bmatrix}$ . Any vector  $x$  which lies in the upper infinite frequency eigensubspace of  $M(s)$  satisfies  $E_M x \neq 0$ .*

*Proof:* Follows directly from (2.26). □

## 2.4 Computational Aspects

There are many software packages currently available for determining the zero structure of a given matrix pencil; one example is the EIGEN or QZ functions available in MATLAB [39]. These packages work quite well when all the zeros of the relevant pencil are finite but are generally not well suited to the task of determining the infinite eigenstructure of matrix pencils.

This is perhaps best performed by using procedures based on algorithms such as the “pencil algorithm” described by Van Dooren [12]. The pencil algorithm employs unitary operations to transform the matrix pencil to block triangular form. The relevant eigensubspace can then be determined from the orthogonal transformation matrices (see following Chapter for an example). It should be noted, that the pencil algorithm is quite expensive; it is generally more economical to determine the finite eigen-structure by the QZ algorithm [10]. However, Van Dooren has indicated that the QZ algorithm could destroy the infinite frequency structure of the pertinent matrix pencil. He therefore suggests that the infinite frequency structure be extracted before application of the QZ algorithm to determine the finite frequency structure.

## Chapter 3

# The Cheap LQ State-Feedback Problem

### 3.1 Introduction

In this chapter we consider the LQ problem for plants with transmission zeros on  $\mathcal{C}^{0e}$ . Specifically, we consider the system

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{bmatrix} A & B \\ C & D(\epsilon) \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \quad x \in \mathbb{R}^n, x(0) = x_0, \quad u \in \mathbb{R}^r, \quad y \in \mathbb{R}^m \quad (3.1)$$

where  $\{A, B\}$  is stabilizable. We assume that  $\exists \epsilon^* > 0$  such that the following hold:

**A3.1** The transfer function

$$G(\epsilon, s) := C(sI - A)^{-1}B + D(\epsilon) \quad (3.2)$$

has no zeros on  $\mathcal{C}^{0e} \forall \epsilon \in (0, \epsilon^*]$ .

**A3.2**  $D(\epsilon)$  is a continuous map on  $(0, \epsilon^*]$ .

**A3.3**  $D(\epsilon)^T D(\epsilon)$  is monotone decreasing on  $[0, \epsilon^*]$ .



For this system we wish to determine the optimal control which minimizes for  $\epsilon \rightarrow 0$

$$J(\epsilon, x_0, u) := \int_0^\infty \|y(t)\|^2 dt < \infty \quad (3.3)$$

such that the stability constraint  $\lim_{t \rightarrow \infty} x(t) = 0 \forall x_0 \in \mathfrak{R}^n$  holds. We note that this includes the usual LQ cost summation

$$J = \int_0^\infty x'(t)Qx(t) + u'(t)Ru(t)dt \quad (3.4)$$

where  $Q = Q' \geq 0$  and  $R = R' > 0$ ; to see this, take, for example,  $C = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}$  and

$$D(\epsilon) = \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix} \text{ in (3.1).}$$

We shall primarily be concerned with the case where  $G(0, s)$  has transmission zeros on  $\mathcal{C}^{0e}$  ( in this case our analysis will be seen to embody the cheap control problem [59, 65, 23, 15]); however, our results will be directly applicable to the case where this is not so.

The motivation for studying these problems is as follows. Consider the system

$$\begin{pmatrix} \dot{x} \\ v \end{pmatrix} = \begin{bmatrix} A & B \\ H & D \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \quad (3.5)$$

It is well known that no acceptable solution (i. e. one which is a realizable stabilizing feedback control with no pole/zero cancellations on  $\Re(s) \geq 0$ ) exists to the LQ problem

$$\min_u \{J(x(0), u) := \int_0^\infty \|v(t)\|^2 dt\} \quad (3.6)$$

when  $G(s) := C(sI - A)^{-1}B + D$  has transmission zeros on  $\mathcal{C}^{0e}$ . In such a case, one alternative which is often pursued is to solve a nearby problem derived from a suitable perturbation of the plant  $G(s)$ . For example, one may use the embedding

$$G(\epsilon, s) = \begin{bmatrix} G(s) \\ \epsilon I \end{bmatrix} \quad (3.7)$$

which has no zeros anywhere. Note that  $G(\epsilon, s)$  is now of the form (3.1) with  $C = \begin{bmatrix} H \\ 0 \end{bmatrix}$  and  $D(\epsilon) = \begin{bmatrix} D \\ \epsilon I \end{bmatrix}$ . Moreover, we have that

$$J(\epsilon, x_0, u) = \int_0^\infty \|v(t)\|^2 dt \xrightarrow{\epsilon \rightarrow 0} J(x_0, u) \quad (3.8)$$

Note that the  $\mathcal{C}^{0e}$  zeros of  $G(\epsilon, s)$  disappear for arbitrarily small perturbations,  $\epsilon$ ; this lends justification to the practice of calling the LQ problem (3.3 singular if  $G(s)$  has transmission zeros on  $\mathcal{C}^{0e}$ .

Implicit in (3.3) above is the requirement that  $y(t) \in \mathcal{L}_2 \forall x(0) \in \mathfrak{R}^n$  and for  $\epsilon \rightarrow 0$ . This automatically excludes any  $y(t)$  for which the corresponding Laplace transform has poles on  $\mathcal{C}^{0e}$ . In particular, this excludes impulsive type signals. Such solutions may occur when  $D(\epsilon)$  loses column rank and are handled elsewhere by the application of distribution theory (see [59], for example). As we shall show, this difficulty may be altogether avoided by analyzing the problem in the frequency domain. To this end, we note that by Parseval's theorem, the cost function (3.3) is also given by

$$J(\epsilon, x_0, u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|y(j\omega)\|^2 d\omega \quad (3.9)$$

In what follows, we carry out the necessary analysis by considering the dependence on  $\epsilon$  of the eigenstructure of the Hamiltonian matrix associated with the problem (3.3). By application of a theorem on the invariant subspaces of analytic matrix functions of a single variable [32], we obtain the required characterization of the limiting solution to (3.3). To the best of our knowledge, this is the first time that this particular “modal” approach to the problem has been taken. The analysis is somewhat in the vein of the “multivariable asymptotic root locus” analysis carried out by Kwakernaak [30] and others; however, this approach leads to what we believe is a much more complete specification of the solution to the problem. For example, the solution derived clearly distinguishes the singular arcs [1] and impulsive components in the control when  $G(0, s)$  has zeros at infinity.

## 3.2 A Generalized Eigenproblem Framework for the Cheap LQ Problem

Recall that given any  $\epsilon$  a solution to (3.3) exists if and only if there exists a positive semi-definite solution  $P(\epsilon)$  to the Riccati equation (see, for example, [31, Theorem 3.4]):

$$A^T P(\epsilon) + P(\epsilon)A + C^T C - (C^T D(\epsilon) + P(\epsilon)B)(D^T(\epsilon)D(\epsilon))^{-1}(D^T(\epsilon)C + B^T P(\epsilon)) = 0 \quad (3.10)$$

The solution may then take the form of the feedback control

$$u(s) = -F_o x(s) \quad (3.11)$$

$$F_o = (D(\epsilon)^T D(\epsilon))^{-1} (B^T P(\epsilon) + D(\epsilon)^T C) \quad (3.12)$$

Alternatively, the optimal state  $x(s)$  and control  $u(s)$  may be directly obtained from the stable trajectories of the autonomous system

$$(-sE_b + A_b(\epsilon)) \begin{pmatrix} x(s) \\ \phi(s) \\ u(s) \end{pmatrix} = E_b \begin{pmatrix} x(0) \\ \phi(0) \\ u(0) \end{pmatrix} \quad (3.13)$$

where

$$E_b := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_b(\epsilon) := \begin{bmatrix} A & 0 & B \\ -C^T C & -A^T & -C^T D(\epsilon) \\ D(\epsilon)^T C & B^T & D(\epsilon)^T D(\epsilon) \end{bmatrix} \quad (3.14)$$

In fact we have the following Lemma:

**Lemma 3.1** *The pair  $(x(s), u(s))$  solves the optimal control problem (3.3) corresponding to the initial condition  $x(0)$  iff  $\begin{pmatrix} x(s) \\ \phi(s) \\ u(s) \end{pmatrix}$  is an asymptotically stable solution of (3.13) for some  $\phi(s)$ .*

*Proof:* Necessity follows from the application of the Pontryagin Maximum Principle (see [1], for example). Sufficiency follows by applying a “completion of the squares” argument similar to that used in [35].

□

Now let  $X_f(\epsilon), \Phi(\epsilon) \in \mathfrak{R}^{n \times n}$  and  $U_f(\epsilon) \in \mathfrak{R}^{r \times n}$  be such that  $\mathcal{S}_-(\epsilon) = \text{Im} \begin{pmatrix} X(\epsilon) \\ \Phi(\epsilon) \\ U(\epsilon) \end{pmatrix}$  where  $\mathcal{S}_-(\epsilon)$  is the stable eigenspace of  $-sE_b + A_b$ . Then if there exists a solution  $\forall x_0 \in \mathfrak{R}^n \Rightarrow X(\epsilon)$  is nonsingular. Moreover,

$$P(\epsilon) = \Phi(\epsilon)X(\epsilon)^{-1} \quad (3.15)$$

and

$$F_0 = -U(\epsilon)X(\epsilon)^{-1} \quad (3.16)$$

We shall have use for some other properties of (3.13) which we state in the next lemma. For generality we temporarily assume that

$$A_b = \begin{bmatrix} A & R & B \\ -C^T C & -A^T & -C^T D(\epsilon) \\ D(\epsilon)^T C & B^T & D(\epsilon)^T D(\epsilon) \end{bmatrix} \quad (3.17)$$

where  $R$  is symmetric positive semi-definite. This will be of importance when we study  $H^\infty$  problems. We define the system matrix

$$M_G(s) := \left[ \begin{array}{c|c} -sI + A & B \\ \hline C & D(0) \end{array} \right] \quad (3.18)$$

**Lemma 3.2** *Let  $W(\epsilon, s) = -sE_b + A_b(\epsilon)$  where  $A_b$  is as given in (3.17). Then*

- (i) *The finite zeros of  $W(\epsilon, s)$  are symmetrically placed about the  $j\omega$ -axis.*
- (ii) *The pencil  $W(\epsilon, s)$  has  $r$  grade 1 infinite eigenvectors  $\forall \epsilon$ , where  $r$  is, as before, the column dimension of  $B$ . Moreover, for  $\epsilon \neq 0$  the associated chains are all of unit length.*

(iii) If  $M_G(s)$  has a complete chain  $\left\{ \begin{pmatrix} x^{(1)} \\ u^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} x^{(k)} \\ u^{(k)} \end{pmatrix} \right\}$  at  $s = \lambda \in \mathcal{C} \cup \infty$ ,

then the vectors  $\left\{ \begin{pmatrix} x^{(1)} \\ 0 \\ u^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} x^{(k)} \\ 0 \\ u^{(k)} \end{pmatrix} \right\}$  forms a partial chain at  $s = \lambda$  for the pencil  $W(0, s)$ .

*Proof:* Define

$$T = \text{diag}\{J, I_r\} \quad (3.19)$$

where  $J$  is the symplectic matrix  $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$ . Then by direct calculation, we observe that

$$TW(\epsilon, s)T = W(\epsilon, s)^*, \quad W(\epsilon, s)^* := W(\epsilon, -s)^T \quad (3.20)$$

Since  $T$  is nonsingular, Part (i) then follows from the fact that the set of finite zeros of  $W(\epsilon, s)$  and  $W(\epsilon, s)^*$  coincide.

Part (ii) follows from the fact that the columns of the matrix  $\begin{bmatrix} 0 \\ 0 \\ I_r \end{bmatrix}$  form a basis for  $\text{Ker } E_b$  and hence may be taken as the full complement of grade 1 eigenvectors of  $-sE_b + A(\epsilon) \forall \epsilon$ . Moreover, it is clear that for  $\epsilon \neq 0$  the recursion (2.26) cannot proceed beyond the first grade.

For part (iii) assume without loss of generality that  $\lambda \in \mathcal{C}$ . Let

$$X = \begin{bmatrix} x^{(1)} & \dots & x^{(k)} \end{bmatrix}, \quad U = \begin{bmatrix} u^{(1)} & \dots & u^{(k)} \end{bmatrix}$$

. Then it is readily seen by substitution in (2.22) that if the columns of  $\begin{bmatrix} X \\ U \end{bmatrix}$  form a basis for the finite eigenspace of  $M_G(s)$  corresponding to some  $\Lambda_f \Rightarrow$  the columns of  $\begin{bmatrix} X \\ 0 \\ U \end{bmatrix}$  form a basis for the finite eigenspace of  $W(0, s)$  corresponding to  $\Lambda_f$ . A similar argument holds for  $\lambda = \infty$ .  $\square$

**Lemma 3.3** *Let the columns of the matrix  $\begin{bmatrix} X \\ \Phi \\ U \end{bmatrix}$  form a basis for the stable eigenspace of the pencil  $W(0, s)$  and the columns of  $\begin{bmatrix} X_\infty \\ U_\infty \end{bmatrix}$  form a basis for the infinite frequency eigenspace of  $M_G(s)$  defined in (3.18).*

*Then*

$$\Phi^T X_\infty = 0 \quad (3.21)$$

*Proof:* From Lemma 3.2 the columns of  $\begin{bmatrix} X_\infty \\ 0 \\ U_\infty \end{bmatrix}$  span a subspace of the infinite frequency eigenspace of  $W(0, s)$ . Moreover, it is easy to show that the columns of  $\begin{bmatrix} \Phi \\ -X \\ U \end{bmatrix}$  spans the stable left eigenspace of  $W(0, s)$ . The result then follows from Lemma 2.5.  $\square$

**Lemma 3.4** *Assume that A3.3 holds. Let  $(x(t), u(t))$  be a vector pair such that*

$$J(\epsilon^*, x_0, u) < \infty \quad (3.22)$$

for some  $\epsilon^* > 0$ . Then the function  $J(\epsilon, x_0, u)$  is monotone decreasing on  $[0, \epsilon^*]$ .

*Proof:* First note that

$$\|y(t)\|^2 = \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{bmatrix} C^T C & C^T D(\epsilon) \\ D^T(\epsilon) C & D^T(\epsilon) D(\epsilon) \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}. \quad (3.23)$$

Assumption A3.3 implies that  $D(\epsilon)D^T(\epsilon)$  is monotone decreasing. Now we use the fact that the eigenvalues of  $\begin{bmatrix} C^T C & C^T D(\epsilon) \\ D^T(\epsilon) C & D^T(\epsilon) D(\epsilon) \end{bmatrix}$  and  $\begin{bmatrix} C & D(\epsilon) \end{bmatrix} \begin{bmatrix} C & D(\epsilon) \end{bmatrix}^T$  agree, except for a set of zero eigenvalues. Hence we have that for  $\epsilon_1 > \epsilon_2$

$$\begin{bmatrix} C & D(\epsilon_1) \end{bmatrix} \begin{bmatrix} C & D(\epsilon_1) \end{bmatrix}^T \geq \begin{bmatrix} C & D(\epsilon_2) \end{bmatrix} \begin{bmatrix} C & D(\epsilon_2) \end{bmatrix}^T \quad (3.24)$$

$\Rightarrow$

$$\begin{bmatrix} C^T C & C^T D(\epsilon_1) \\ D^T(\epsilon_1) C & D^T(\epsilon_1) D(\epsilon_1) \end{bmatrix} \geq \begin{bmatrix} C^T C & C^T D(\epsilon_2) \\ D^T(\epsilon_2) C & D^T(\epsilon_2) D(\epsilon_2) \end{bmatrix} \quad (3.25)$$

The proof is completed by noting that the integrand in (3.3) is therefore monotone decreasing in  $\epsilon$ . □

**Lemma 3.5** *There exists a solution,  $(x(t), u(t))$ , to the problem (3.3) at  $\epsilon = 0$  if there exists a solution,  $(x^*(t), u^*(t))$ , to the problem at  $\epsilon = \epsilon^*$  for some  $\epsilon^* > 0$ .*

*Proof:* By Assumption A3.3 and Lemma 3.4 (3.3) is monotone decreasing in  $\epsilon$ . Since this function is non-negative, and since, by hypothesis,  $J(\epsilon^*, x^*(0), u^*)$  exists, it follows that  $J(\epsilon, x^*(0), u^*)$  is a bounded monotone decreasing function of  $\epsilon$  and hence converges as  $\epsilon \rightarrow 0$ . This implies the existence of a solution at  $\epsilon = 0$ . □



### 3.3 Solution to the Cheap LQ State-Feedback

#### Problem

Our first main result of the chapter follows:

**Theorem 3.1** *Assume that A3.1-3.3 hold. Let  $\begin{bmatrix} X_{0e}^{(l)} \\ U_{0e}^{(l)} \end{bmatrix}$  be a matrix whose columns form a basis for the sum of the finite  $C^{0e}$  eigenspace and the lower infinite frequency eigensubspace of the pencil  $M_G(s)$ . Then it holds that*

$$\lim_{\epsilon \rightarrow 0} \mathcal{S}_-(\epsilon) = \mathcal{S}_-(0) \cup \text{Im} \left( \begin{bmatrix} X_{0e}^{(l)} \\ 0 \\ U_{0e}^{(l)} \end{bmatrix} \right) \quad (3.26)$$

In order to prove this theorem we will need two supporting lemmas. The first is proved in Gohberg *et al.* [20]:

**Lemma 3.6** *Let  $A(\mu), \mu \in \Omega$  be an  $n \times n$  complex matrix-valued function analytic in a domain  $\Omega$  of  $\mathcal{C}$ . Let  $r = \max \text{rank } A(\mu)$ . Then there exist  $n$ -dimensional vector valued functions  $y_i(\mu), i \in \underline{n}$  such that*

(i)  $y_i(\mu)$  is analytic in  $\Omega \forall i \in \underline{n}$ ,

(ii)  $y_1(\mu), \dots, y_r(\mu)$  are linearly independent  $\forall \mu \in \Omega$ ,

(iii)  $y_{r+1}(\mu), \dots, y_n(\mu)$  are linearly independent  $\forall \mu \in \Omega$ ,

(iv)  $\text{Span}\{y_1(\mu), \dots, y_r(\mu)\} = \text{Im } A(\mu)$  and  $\text{Span}\{y_{r+1}(\mu), \dots, y_n(\mu)\} = \text{Ker } A(\mu)$

$\forall \mu \in \Omega$ , except for those isolated points,  $\mu_l \in \Omega$ , for which  $\text{rank } A(\mu_l) < r$ . At these points the following hold

$$\text{Span}\{y_1(\mu_l), \dots, y_r(\mu_l)\} \supset \text{Im } A(\mu_l) \quad (3.27)$$

$$\text{Span}\{y_{r+1}(\mu_l), \dots, y_n(\mu_l)\} \subset \text{Ker } A(\mu_l) \quad (3.28)$$

*Proof:* See Chapter S6 of [20]. □

The next lemma is easily proved using standard matrix theory:

**Lemma 3.7** *Suppose  $T \in \mathbb{C}^{n \times n}$  has  $k$  eigenvalues at  $s = \lambda$  of respective orders  $\alpha_i$ ,  $i \in \underline{k}$ . Define  $\alpha := \sum_{i \in \underline{k}} \alpha_i$ . Then it holds that*

$$\dim \text{Ker } (T - \lambda I)^q \geq q \quad (3.29)$$

$\forall q \in \underline{\alpha}$ .

*Proof of Theorem 3.1:* Assume that  $M_G(s)$  has  $k$  distinct  $\mathbb{C}^{0e}$  eigenvalues of various orders. Consider the neighborhood  $\Omega_{\epsilon^*} := (-\epsilon^*, \epsilon^*)$  of  $\epsilon = 0$ . Let  $\alpha \in \mathfrak{R}$  be such that on  $\Omega_{\epsilon^*}$   $W(\epsilon, s)$  has no eigenvalue  $\lambda = -\alpha$ . Now ;make the transformation

$$\tilde{s} = s + \alpha \quad (3.30)$$

and note that the eigenstructure of  $W(\epsilon, s)$  at  $s = \lambda$  is isomorphic to that of  $W(\epsilon, \tilde{s}) = -\tilde{s}E_b + (\alpha E_b + A_b(\epsilon))$  at  $\tilde{s} = \lambda + \alpha$ . Also note that by construction  $\alpha E_b + A_b(\epsilon)$  is nonsingular on  $\Omega_{\epsilon^*}$  and define

$$\Theta(\epsilon) = (\alpha E_b + A_b(\epsilon))^{-1} E_b \quad (3.31)$$

Now the right eigenstructure of  $W(\epsilon, \tilde{s})$  is identical to that of

$$(\alpha E_b + A_b(\epsilon))^{-1} W(\epsilon, \tilde{s}) = -(\alpha E_b + A_b(\epsilon))^{-1} E_b \tilde{s} + I \quad (3.32)$$

It therefore follows that the eigenstructure of the pencil  $W(\epsilon, s)$  at  $s = \lambda$  is now isomorphic to that of the matrix  $\Theta(\epsilon)$  at  $\rho = \frac{1}{\lambda + \alpha}$ . Figure 3.1 shows how the conformal map  $\rho = \frac{1}{s + \alpha}$  maps the  $s$ -plane; note that the left half  $s$ -plane is mapped onto the exterior  $\mathcal{J}_0$  of the circle  $\mathcal{J}$ .

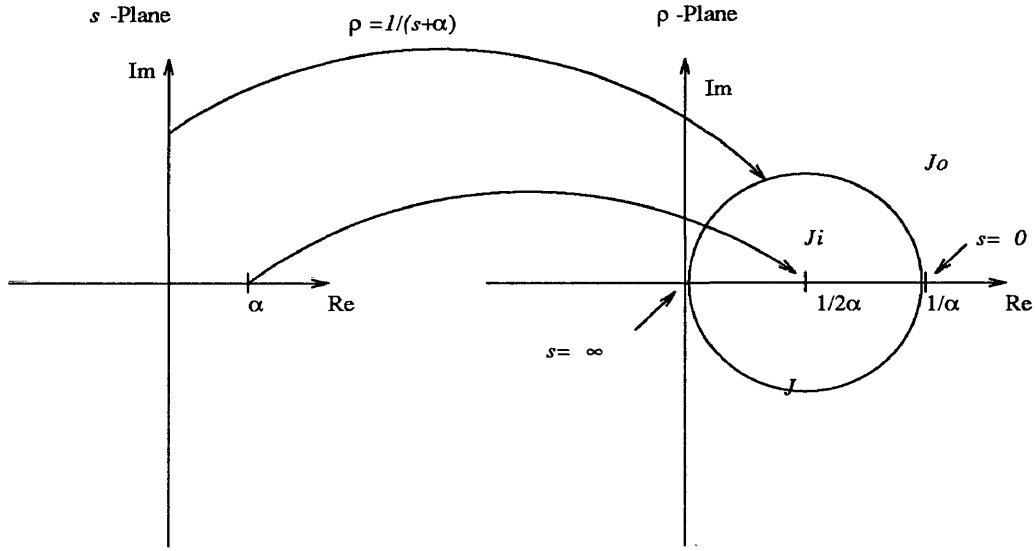


Figure 3.1: Mapping of the  $s$ -Plane under  $\rho = \frac{1}{s+\alpha}$

From Lemma 3.2 we know that  $\forall \epsilon \Theta(\epsilon)$  has a characteristic polynomial of the form

$$\pi(\epsilon, \rho) = \rho^r \pi_o(\epsilon, \rho) \pi_i(\epsilon, \rho) \quad (3.33)$$

where the zeros of  $\pi_o$  are all in  $\mathcal{J}_o$  for  $\epsilon \neq 0$ . Without loss of generality, we assume that  $\epsilon^*$  is small enough so that those eigenvalues in  $\mathcal{J}_o$  which do approach  $\mathcal{J}$  as  $\epsilon \rightarrow 0$  are distinct from those remaining in  $\mathcal{J}_o \forall \epsilon \in \Omega_{\epsilon^*}$ . We also assume that  $\forall \epsilon \in \Omega_{\epsilon^*}$  those eigenvalues in  $\mathcal{J}_o$  which approach  $\rho = \rho_i$  on  $\mathcal{J}$  are distinct from those approaching the value  $\rho = \rho_j$  on  $\mathcal{J}$ ,  $i \neq j$ . Thus, on  $\Omega_{\epsilon^*}$  we have the coprime factorization :

$$\pi_o = \pi_1 \pi_2, \quad \pi_2 := \pi_{2_1} \cdots \pi_{2_k}$$

where the zeros of  $\pi_1$  are in  $\mathcal{J}_o \forall \epsilon \in \Omega_{\epsilon^*}$ , and the zeros of  $\pi_{2_i}$ ,  $i \in \underline{k}$ , approach the value  $\rho = \rho_i$  on  $\mathcal{J}$  as  $\epsilon \rightarrow 0$ . Note that  $\Theta(\epsilon)$  is analytic in  $\Omega_{\epsilon^*}$  by virtue of

the nonsingularity of  $\alpha E_b + A_b(\epsilon)$ ; hence the polynomials  $\pi_1$  and  $\pi_{2_i}$ ,  $i \in \underline{k}$ , are continuous in  $\epsilon$  on  $\Omega^*$ .

Let  $\delta$  be the order of  $\pi_1$  and  $\delta_i$  the order of  $\pi_{2_i}$ ,  $i \in \underline{k}$ . Now,  $\forall \epsilon \in \Omega_{\epsilon^*}$  we have that

$$\mathcal{S}_-(\epsilon) = \text{Ker } \pi_o(\epsilon, \Theta(\epsilon)) = \text{Ker } \pi_1(\epsilon, \Theta(\epsilon)) \oplus \text{Ker } \pi_{2_1}(\epsilon, \Theta(\epsilon)) \oplus \cdots \oplus \text{Ker } \pi_{2_k}(\epsilon, \Theta(\epsilon))$$

the latter equality following from the coprimeness of the respective polynomials.

Consider, first, the subspace  $\text{Ker } \pi_1(\epsilon, \Theta(\epsilon))$ ; since the dimension of this subspace is constant ( $= \delta$ ) on  $\Omega_{\epsilon^*}$  it follows from Lemma 3.6 that

$$\lim_{\epsilon \rightarrow 0} \text{Ker } \pi_1(\epsilon, \Theta(\epsilon)) = \text{Ker } \pi_1(0, \Theta(0)) \quad (3.34)$$

Next consider the subspace  $\text{Ker } \pi_{2_i}(\epsilon, \Theta(\epsilon))$ . By Lemma 3.2 the chains corresponding to  $\rho = \rho_i$  have a cumulative length  $l \geq \delta_i$ . It therefore follows from Lemma 3.7 that  $\dim \text{Ker } \pi_{2_i}(0, \Theta(0)) = \dim \text{Ker } (\Theta(0) - \rho_i I)^{\delta_i} \geq \delta_i$ . This proves that if  $\Xi_{oe}(\epsilon)$  is a matrix whose columns form a basis for  $\text{Ker } \pi_2(\epsilon, \Theta(\epsilon))$ ,  $\epsilon \neq 0$ , then by Lemma 3.6  $\Xi_{oe} := \lim_{\epsilon \rightarrow 0} \Xi_{oe}(\epsilon)$  exists and satisfies

$$\text{Im } \Xi_{oe} \subset \text{Ker } (\Theta(0) - \rho_1 I)^{\delta_1} \oplus \cdots \oplus \text{Ker } (\Theta(0) - \rho_k I)^{\delta_k} \quad (3.35)$$

Summarizing, we have that

$$\lim_{\epsilon \rightarrow 0} \mathcal{S}_-(\epsilon) = \mathcal{S}_-(0) \cup \text{Im } \Xi_{oe} \quad (3.36)$$

where  $\text{rank } \Xi_{oe} = \sum_{i=1}^k \delta_i = \text{rank } \Xi_{oe}(\epsilon)$  for any  $\epsilon \in \Omega_{\epsilon^*}$ .

The proof is now almost complete save for the characterization of  $\Xi_{oe}$ . To obtain this, we note that the eigenspace of  $W(0, s)$  associated with the  $\mathcal{C}^{0e}$  eigenvalues

determines those solutions of (3.13) with poles on the  $j\omega$ -axis or at infinity ( polynomial solutions). It follows that the corresponding cost (3.9) would be infinite unless

all such solutions lie in  $\text{Ker} \begin{bmatrix} C & D \end{bmatrix}$  (in which case the cost is zero). Now note

that by Lemma 3.2 the columns of  $\begin{bmatrix} X_{0\epsilon}^{(l)} \\ 0 \\ U_{0\epsilon}^{(l)} \end{bmatrix}$  do indeed form partial chains of  $W(\epsilon, s)$

where  $\begin{bmatrix} X_{0\epsilon}^{(l)} \\ U_{0\epsilon}^{(l)} \end{bmatrix} \subset \text{Ker} \begin{bmatrix} C & D \end{bmatrix}$ ; moreover, the choice of any other vectors in the respective chains results in infinite cost in (3.3).  $\square$

Theorem 3.1 verifies that the limiting optimal solution is indeed determined by the modal responses of  $W(0, s)$  as suggested by application of the Minimum Principle at  $\epsilon = 0$ . The complexity in deriving the limiting subspace when  $W(0, s)$  has infinite eigenvalues arises from the fact that while the relevant initial conditions may depend on the highest grade eigenvector of  $M_G(s)$ , the resulting solutions are spanned by eigenvectors of lower grade. This results in a limiting feedback gain  $F_0$  (3.12) which is infinite.

Implicit in the above is the assumption that a solution exists at  $\epsilon = 0$ . The following lemma is helpful in determining when a solution to the problem actually exists.

**Lemma 3.8** Let  $\mathcal{S}_-(0) = \text{Im} \begin{bmatrix} X \\ \Phi \\ U \end{bmatrix}$  and let  $\begin{bmatrix} X_{oe}^{(u)} \\ U_{oe}^{(u)} \end{bmatrix}$  be a matrix whose columns form a basis for the sum of the finite  $\mathcal{C}^{0e}$  eigenspace and the upper infinite frequency eigensubspace of  $M_G(s)$ . Define  $\mathcal{X} = \begin{bmatrix} X & X_{oe}^{(u)} \end{bmatrix}$ . Then

(i)  $\text{Im } \mathcal{X}$  is the space of initial conditions  $x_0$  for which the cost (3.3) is finite.

(ii)  $\text{rank } \mathcal{X} < n \Leftrightarrow$  no solution to the problem (3.3) exists.

*Proof:* Part (i) follows from the arguments given in the proof to Theorem 3.1 and the discussions of Chapter 2. For part (ii) note that if the hypothesis holds, then since the set of eigenvectors of  $W(0, s)$  are linearly independent  $\Rightarrow$  the existence of initial conditions  $x_0$  which can be associated only with the unstable eigenspace or the space of  $\mathcal{C}^{0e}$  eigenvectors of higher grade  $\Rightarrow$  no solution exists. On the other hand, if  $\text{rank } \mathcal{X} = n \Rightarrow \forall x_0 \in \mathfrak{R}^n$  there exists a corresponding trajectory  $\begin{pmatrix} x(s) \\ \phi(s) \\ u(s) \end{pmatrix}$  in (3.13) such that  $\lim_{t \rightarrow \infty} x(t) = 0$  and (3.9) is bounded  $\Rightarrow$  a solution exists.  $\square$

The following theorem gives us a formula for the limiting value of (3.3).

**Theorem 3.2** Let  $\begin{bmatrix} X \\ \Phi \\ U \end{bmatrix}$  and  $\begin{bmatrix} X_{oe}^{(u)} \\ U_{oe}^{(u)} \end{bmatrix}$  be defined as in the previous lemma. Assume that there exists a solution to the cheap control problem (3.3). Then the limiting cost is given by

$$J(x_0) := J(0, x_0, u_*) = x_0^T P_* x_0 \quad (3.37)$$

where  $u_*(s)$  is the optimal control and  $P_*$  is a symmetric positive semi-definite  $n \times n$  matrix given by

$$P_* = \Phi H^T \quad (3.38)$$

where  $H$  is a partition of the nonsingular matrix  $\begin{bmatrix} H & H_{oe} \end{bmatrix}$  which satisfies

$$\begin{bmatrix} H^T \\ H_{oe}^T \end{bmatrix} \begin{bmatrix} X & X_{oe}^{(u)} \end{bmatrix} = I \quad (3.39)$$

*Proof:* By Lemma 3.8, the existence of a solution to the problem (3.3) implies that  $\forall x_0 \in \mathfrak{R}^n \exists$  vectors  $v, v_{oe}$  such that

$$x_0 = Xv + X_{oe}^{(u)}v_{oe} \quad (3.40)$$

Moreover, the resulting cost is equivalent to that obtained for  $x_0 = Xv$  since the cost pertaining to the  $C^{oe}$  eigenspace of  $W(0, s)$  is zero. In the time domain the optimal state and control trajectories corresponding to  $x_0$  is given by  $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} X \\ \Phi \end{bmatrix} e^{\Lambda t} v$

for  $t > 0$ , where  $\Lambda$  is strictly Hurwitz and satisfies

$$\begin{bmatrix} A & 0 & B \\ -C^T C & -A^T & -C^T D \\ D^T C & B^T & D^T D \end{bmatrix} \begin{bmatrix} X \\ \Phi \\ U \end{bmatrix} = \begin{bmatrix} X \\ \Phi \\ 0 \end{bmatrix} \Lambda \quad (3.41)$$

The optimal cost is thus given by

$$\begin{aligned} J(x_0, u) &= \int_0^\infty \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} dt \\ &= v^T \left( \int_0^\infty e^{\Lambda^T t} \begin{bmatrix} X \\ U \end{bmatrix}^T \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} e^{\Lambda t} dt \right) v \end{aligned} \quad (3.42)$$

Now from (3.41) it can be shown that

$$\begin{bmatrix} X \\ U \end{bmatrix}^T \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} = -(X^T \Phi \Lambda + \Lambda^T X^T \Phi) \quad (3.43)$$

Hence we have that

$$\begin{aligned} J(x_0, u) &= -v^T \left( \int_0^\infty e^{\Lambda^T t} (X^T \Phi \Lambda + \Lambda^T X^T \Phi) e^{\Lambda t} dt \right) v \\ &= -v^T \left( \int_0^\infty \frac{d}{dt} (e^{\Lambda^T t} X^T \Phi e^{\Lambda t}) dt \right) v \\ &= v^T X^T \Phi v \end{aligned} \quad (3.44)$$

the last equality following from the fact that  $\Lambda$  is Hurwitz. Since  $\begin{bmatrix} X & X_{0e}^{(u)} \end{bmatrix}$  is nonsingular  $\Rightarrow \exists$  a matrix  $\begin{bmatrix} H & H_{oe} \end{bmatrix}$  such that

$$\begin{bmatrix} H^T \\ H_{oe}^T \end{bmatrix} \begin{bmatrix} X & X_{0e}^{(u)} \end{bmatrix} = I_n. \quad (3.45)$$

Left multiply (3.40) by  $H^T$  to obtain

$$v = H^T x_0 \quad (3.46)$$

and substitute in (3.44) to get

$$J(x_0, u) = x_0^T H X^T \Phi H^T x_0 \quad (3.47)$$

Now we note that from Lemma 3.3 that  $X_{oe}^T \Phi = 0 \Rightarrow H_{oe} X_{oe}^T \Phi = 0 \Rightarrow$

$$(H X^T - I) \Phi = 0 \quad (3.48)$$

the last relationship following from (3.45). Rewrite (3.48) as  $H X^T \Phi = \Phi$  and substitute in (3.47) to give

$$\begin{aligned} J &= x_0^T \Phi H^T x_0 \\ &= x_0^T P_* x_0 \end{aligned} \quad (3.49)$$



The equations above show that  $P_*$  is indeed a matrix of a positive semi-definite form. To show that it is also symmetric define  $N := \Phi^T X - X^T \Phi$  and pre-multiply both sides of (3.41) by  $\begin{bmatrix} \Phi^T & -X^T & 0 \end{bmatrix}$  to obtain

$$N\Lambda = \Phi^T AX + (\Phi^T AX)^T + X^T C^T CX - U^T D^T DU. \quad (3.50)$$

Since the RHS of (3.50) is symmetric and  $N$  is skew symmetric, it follows that

$$(-\Lambda^T)N - N\Lambda = 0 \quad (3.51)$$

Since  $\Lambda$  is Hurwitz, and the eigenvalues of  $-\Lambda^T$  and  $\Lambda$  are disjoint  $\Rightarrow$  by Lemma 1.5 [31] that the unique solution to the above is  $N = 0 \Rightarrow \Phi^T X = X^T \Phi$ .

□

To complete this section we include a simple example taken from [1, pg. 248].

Here, we have  $G(s) \stackrel{s}{=} M_G(s) = \left[ \begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & -1 \\ \hline 1 & 0 & 0 \end{array} \right]$ .  $W(0, s)$  has a finite LHP zero at

$s = -1$  with corresponding eigenvector  $(-2 \ 1 \ 2 \ 2 \ 1)^T$ . The infinite eigenvectors for

$M_G(s)$  are the columns of  $\left[ \begin{array}{cc|c} 0 & 1 & \\ \hline 0 & -1 & \\ 1 & 0 & \end{array} \right]$ . Hence, corresponding to an initial condition

$k(1 \ -1)^T$  the optimal input is  $-k\delta(t)$ . For initial conditions along  $(-2 \ 1)^T$  the

required input is of the form  $ke^{-t}$ ; this vector is indeed the “singular arc” for the

infinite horizon problem. These results may be compared with those of [1].

### 3.4 Summary

In this chapter we have characterized the limiting solution to cheap LQ problems by applying a result on the invariant subspaces of an analytic matrix function of a scalar parameter  $\epsilon$  [20]. The proof of the result has been facilitated by the use of the generalized eigenproblem formulation of the usual Riccati equation. A distinct advantage of the approach is that we can directly determine the limiting cost from the system parameters (Theorem 3.2).

Our results show that when the plant has  $\mathcal{C}^{0e}$  zeros, the solution is determined by the stable eigenspace of a particular matrix pencil,  $W(0, s)$ , and the  $\mathcal{C}^{0e}$  zero vectors of the plant. This is in keeping with known results for this problem [59, 30]. These results will now be applied to the cheap LQG problem.

## Chapter 4

### The Cheap LQG Problem

#### 4.1 Introduction

In what follows the results of Chapter 3 are applied to obtain a characterization for the limiting form of stochastic regulators. Specifically, we consider the system

$$\begin{pmatrix} \dot{x} \\ y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12}(\epsilon) \\ C_2 & D_{21}(\epsilon) & D_{22} \end{bmatrix} \begin{pmatrix} x \\ u_1 \\ u_2 \end{pmatrix} \quad (4.1)$$

where  $y_i(t) \in \mathbb{R}^{m_i}$  and  $u_i(t) \in \mathbb{R}^{r_i}$ ,  $i \in \underline{2}$ . We assume that  $r_1 \leq m_1$  and  $m_2 \leq r_2$  and that  $\{A, B_2\}$  is stabilizable and  $\{C_2, A\}$  is detectable. Without loss of generality, we assume that

$$D_{11} = 0, \quad D_{22} = 0. \quad (4.2)$$

If  $D_{22} \neq 0$  then loop-shifting procedures, as outlined in [50], can be used to determine transformations to zero this term. In some cases, a non-zero  $D_{11}$  can also be zeroed

by the methods described in [50]. Note that if no feedback composition results in a non-zero  $D_{11}$ , then the resulting LQG cost is infinite and the problem ill-posed.

For our analysis we require that there exist  $\epsilon^* > 0$  for which the following hold:

**A4.1** The transfer functions  $G_{12}(\epsilon, s)$  and  $G_{21}^T(\epsilon, s)$ , where  $G_{12}(\epsilon, s) := C_1(sI - A)^{-1}B_2 + D_{12}(\epsilon)$  and  $G_{21}^T(\epsilon, s) := C_2(sI - A)^{-1}B_1 + D_{21}(\epsilon)$ , are injective on  $(0, \epsilon^*] \times \mathcal{C}^{0\epsilon}$ .

**A4.2**  $D_{12}(\epsilon)$  and  $D_{21}(\epsilon)^T$  are continuous maps on  $[0, \epsilon^*]$ .

**A4.3**  $D_{12}(\epsilon)^T D_{12}(\epsilon)$  and  $D_{21}(\epsilon) D_{21}^T(\epsilon)$  are monotone decreasing on  $[0, \epsilon^*]$ .

The vector  $u_1$  is assumed to be unit intensity Gaussian white noise. The sub-vector  $u_2$  represents the control inputs and  $y_2$  the measurable outputs. Our objective is now to determine for  $\epsilon \rightarrow 0$  the control  $u_2$  to minimize

$$J(\epsilon) := \mathbb{E}(y_1^T y_1). \quad (4.3)$$

We shall again be primarily concerned with the case where  $G_{21}^T(0, s)$  and/or  $G_{12}(0, s)$  have zeros on  $\mathcal{C}^{0\epsilon}$ . Following the notation in the LQ problem we shall therefore, in this case, refer to the problem (4.3) as a **cheap LQG problem**.

We begin with a brief discussion on system matrices which are analytic in a scalar parameter  $\epsilon$ .

## 4.2 System Matrix Parameterizations of Well-Posed Systems

In this section, we digress momentarily to consider the following problem: Suppose we are given the polynomial matrix

$$M(\epsilon, s) = \left[ \begin{array}{c|c} M_{11}(\epsilon, s) & M_{12}(\epsilon, s) \\ \hline M_{21}(\epsilon, s) & M_{22}(\epsilon, s) \end{array} \right] \quad (4.4)$$

which is polynomial in  $s$  and analytic in  $\epsilon$  for all  $\epsilon$  in some region  $\mathcal{W} \subset \mathfrak{R}$ . Define the mapping

$$G(\epsilon, s) = -M_{21}(\epsilon, s)M_{11}(\epsilon, s)^{-1}M_{12}(\epsilon, s) + M_{22}(\epsilon, s). \quad (4.5)$$

The problem is to determine whether  $\lim_{\epsilon \rightarrow \epsilon_0} G(\epsilon, s)$  exists for some given  $\epsilon_0 \in \mathcal{W}$ .  $M(\epsilon, s)$  can thus be considered as the Rosenbrock system matrix [45], say, of a linear system which is parameterized in terms of a scalar  $\epsilon$ .

Directly related to this issue is the notion of what we term a well-posed or an i/o well-posed system matrix.

**Definition 4.1** *The Rosenbrock system matrix*

$$P(s) = \left[ \begin{array}{c|c} P_{11}(s) & P_{12}(s) \\ \hline P_{21}(s) & P_{22}(s) \end{array} \right] \quad (4.6)$$

*is said to be i/o well posed if given any  $u(s)$  in the field of rational functions, there exists a unique  $y(s)$  satisfying*

$$P_{11}(s)x(s) + P_{12}(s)u(s) = 0 \quad (4.7)$$

$$P_{21}(s)x(s) + P_{22}(s)u(s) = y(s) \quad (4.8)$$

for some rational vector function  $x(s)$ . If  $x(s)$  is also unique for each  $u(s)$ , the system matrix will be said to be **well-posed**.

Equivalently, it is seen that  $P(s)$  is i/o well-posed if and only if there exists a unique function,  $T(s)$ , such that  $y(s) = T(s)u(s)$ , where the pair  $(y(s), u(s))$  satisfy (4.7, 4.8), even when  $P_{11}(s)$  is singular. Further thought shows that  $P(s)$  is i/o well posed if and only if the following hold

- (i)  $\exists$  at least one solution  $x(s)$  to (4.7) for any given  $u(s)$ .
- (ii) If  $x_1(s)$  and  $x_2(s)$  are solutions to (4.7) corresponding to some  $u(s)$  then

$$P_{21}(s)(x_1(s) - x_2(s)) = 0, \text{ a.e.} \quad (4.9)$$

This is embodied in the following lemma:

**Lemma 4.1** *The system matrix  $P(s)$  is i/o well posed iff the following condition holds*

$$\text{rank } P_{11}(s) = \text{rank} \begin{bmatrix} P_{11}(s) & P_{12}(s) \end{bmatrix} \quad (4.10)$$

$$= \text{rank} \begin{bmatrix} P_{11}(s) \\ P_{21}(s) \end{bmatrix} \quad (4.11)$$

for almost all  $s \in \mathcal{C}$ . It is well-posed iff (4.10) holds and  $P_{11}(s)$  is injective almost everywhere in  $\mathcal{C}$ .

*Proof:* From elementary linear algebra (see [58], for example) we know that given any  $u(s)$ , that (4.10) and the injectivity of  $P_{11}(s)$  are necessary and sufficient conditions for the existence of a unique solution  $x(s)$  in (4.7). Note that an equivalent expression for (4.10) is that  $\text{Im } P_{12}u(s) \subset \text{Im } P_{11}(s)$  almost everywhere.

Again, we note that, by itself, (4.10) is necessary and sufficient for the existence of at least one  $x(s)$  given any  $u(s)$ . We also note that (4.11) is equivalent to saying that  $\text{Ker } P_{21}(s) \subset \text{Ker } P_{11}(s)$  almost everywhere. We claim that this is a necessary and sufficient condition for  $y(s)$  to be uniquely determined by (4.8) for all  $x(s)$  which is a solution to (4.7) for a given  $u(s)$ . To see this we recall that any solution  $x(s)$  to (4.7) takes the form

$$x(s) = x_i(s) + x_k(s) \quad (4.12)$$

where  $x_i(s) \in \text{Ker } {}^\perp P_{11}(s)$  and is therefore uniquely determined by  $u(s)$  and  $x_k(s) \in \text{Ker } P_{11}(s)$ . The result then follows by the fact that

$$y(s) = P_{21}x_i(s) + P_{21}x_k(s) + P_{22}u(s) \quad (4.13)$$

and is unique, given  $u(s)$  iff  $P_{21}x_k(s) = 0$ .

□

The following corollary to this Lemma provides us with a means for calculating the transfer function of an i/o well-posed system matrix.

**Corollary 4.1** *If the conditions of Lemma 4.1 hold the corresponding transfer function from  $u(s)$  to  $y(s)$  is given by*

$$G(s) = -P_{21}P_{11}^\dagger P_{12}(s) + P_{22}(s) \quad (4.14)$$

where  $P_{11}^\dagger(s)$  is the pseudo-inverse of  $P_{11}(s)$  evaluated over the field of polynomial functions and defined by

$$x_i(s) = -P_{11}^\dagger P_{12}u(s) \quad (4.15)$$

*Proof:* Follows from Lemma 4.1 and elementary linear algebra considerations.  $\square$

Note that for  $P_{11}(s)$  square the definition of a well-posed system matrix coincides with the orthodox definition. Note also that if  $P(s)$  is well-posed then it is also i/o well-posed.

The motivation for studying this problem arises from the fact that the stochastic regulators which solve the cheap LQG problem will be characterized by system matrices which are analytic functions of the weighting parameter  $\epsilon$ . In general, the exact dependence on  $\epsilon$  will not be known. Since we will be trying to determine the limiting form of these regulators as  $\epsilon \rightarrow 0$ , it is prudent to ask, for example, if  $G(\epsilon_0, s)$  exists even when the limiting system matrix  $M(\epsilon_0, s)$  is not well posed.

To demonstrate this limiting behavior, consider the following example:

$$M(s) = \left[ \begin{array}{cc|c} s\epsilon^2 & s\epsilon & \alpha(\epsilon) \\ s\epsilon & 1 & \beta \\ \hline \gamma & 1 & 0 \end{array} \right] \quad (4.16)$$

For  $\epsilon \neq 0$   $M(\epsilon, s)$  is well-posed with corresponding transfer function

$$G(\epsilon, s) = -\frac{\alpha(\epsilon)(\gamma - \epsilon s) + \beta\epsilon(\epsilon - 1)s}{\epsilon^2 s(1 - s)} \quad (4.17)$$

For  $\epsilon = 0$  the (1,1) block is singular and  $M(0, s)$  may or may not be i/o well-posed.

If it is well posed then the corresponding transfer function may be determined from

(4.14); in this case we have  $P_{11}^\dagger(s) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$  if  $M(0, s)$  is i/o well-posed then the corresponding transfer function is  $G(0, s) = -\beta$ .



Now let us define

$$G_0(s) := \lim_{\epsilon \rightarrow 0} G(\epsilon, s) \quad (4.18)$$

if this limit exists. The well-posedness of this system matrix for  $\epsilon \rightarrow 0$  is summarized in Table 4.1 for various values of  $\alpha$ ,  $\beta$  and  $\gamma$ . Observe, in particular, that the existence of an i/o well-posed limiting system matrix does not guarantee the existence of  $G_0(s)$ .

| $\alpha(\epsilon)$ | $\beta$ | $\gamma$ | i/o well posed limit? | $G(\epsilon, s),$<br>$\epsilon > 0$      | $G(0, s)$   | $G_0(s)$            |
|--------------------|---------|----------|-----------------------|--|-------------|---------------------|
| $\epsilon$         | 0       | 1        | no                    | $\frac{\epsilon s - 1}{\epsilon s(1-s)}$ | not defined | not defined         |
| $\epsilon^2$       | 0       | 1        | no                    | $\frac{\epsilon s - 1}{s(1-s)}$          | not defined | $\frac{-1}{s(1-s)}$ |
| $\epsilon$         | 0       | 0        | yes                   | $\frac{1}{1-s}$                          | 0           | $\frac{1}{1-s}$     |
| $\epsilon$         | 1       | 0        | yes                   | $\frac{s}{\epsilon s(1-s)}$              | -1          | not defined         |

Table 4.1: Examples of Parameterized System Matrices and their Limits

We can obtain considerable insight into the problem by first considering the case where  $M(\epsilon, s)$  is a polynomial matrix in  $\epsilon$ . In this case the required limit exists if and only if the corresponding transfer function  $G(\epsilon, s)$  has no poles at  $\epsilon = \epsilon_0$ . Thus the problem is seen to involve pole/zero cancellations at  $\epsilon = \epsilon_0$ ; the required conditions must be determined from a knowledge of the relevant pole and zero directions at  $\epsilon = \epsilon_0$  [37]. It is important to note that unless  $M(\epsilon, s)$  is linear in  $\epsilon$ , the vectors which specify these pole/zero directions will themselves be polynomial functions of  $\epsilon$ .

In the more general case, where  $M(\epsilon, s)$  is analytic in  $\epsilon$ , we claim that the same type of analysis can still be carried out. To see this, observe that the poles and

zeros of  $G(\epsilon, s)$  are still defined, at least in the scalar case, since  $G(\epsilon, s)$  is a matrix valued meromorphic function<sup>1</sup> in  $\epsilon$  and since the definition of poles and zeros for rational scalar functions are, in fact, directly derived from the relevant definition for scalar meromorphic functions. For matrices, the zeros of  $G(\epsilon, s)$  can be defined as the values of  $\epsilon$  for which  $M(\epsilon, s)$  loses full (column/row) rank. The poles will, as usual, be defined to be the zeros of  $M_{11}(\epsilon, s)$ . It is therefore seen that the definitions of poles and zeros for matrix valued meromorphic functions are natural extensions of the usual definitions for rational transfer function matrices (defined on  $\mathcal{C}$ ) except that now, there will, in general be an infinite number of poles and zeros to be accounted for.

If we now apply the pole/zero cancellation arguments given above, it is seen that the required vectors will be analytic functions of  $\epsilon$  and will be difficult to determine even when the elements of  $M(\epsilon, s)$  are explicitly known. This will not be the case in the study we will soon undertake. Presumably, the required condition can also be determined by deriving a sufficient number of terms in the power series expansion of  $M(\epsilon, s)$  about  $\epsilon = \epsilon_0$ ; however, this may also require an explicit knowledge of  $M(\epsilon, s)$ .

In lieu of a complete description of the solution to the subproblem defined in this section, we state the following lemmas which give sufficient conditions pertaining to

---

<sup>1</sup>Meromorphic functions are functions which can be written as the ratio of two analytic functions. See [46][Ch. 10] for a more formal definition. Rational functions are examples of meromorphic functions.

the existence of the required limit. Both follow from the discussions given above; the first states that the limit exists if  $G(\epsilon, s)$  has no poles at  $\epsilon_0$ . The second states that the limit cannot exist if  $G(\epsilon, s)$  has at least one pole, but no zero at  $\epsilon = \epsilon_0$ .

**Lemma 4.2** *If it holds that  $\det M_{11}(\epsilon_0, s) \neq 0$ , then  $\lim_{\epsilon \rightarrow \epsilon_0} G(\epsilon, s) = G(\epsilon_0, s)$  exists.*

**Lemma 4.3** *If it holds that  $M_{11}(\epsilon_0, s)$  is singular but  $M(\epsilon, s)$  has full column or row rank at  $\epsilon = \epsilon_0$  then  $\lim_{\epsilon \rightarrow \epsilon_0} G(\epsilon, s)$  does not exist.*

### 4.3 The Cheap LQG Problem

In this section, we consider the system (4.1) and apply the results of Chapter 3 to determine the limiting form of the stochastic regulator which minimizes

$$J(\epsilon) = \mathbb{E}(y_1^T y_1) \quad (4.19)$$

as  $\epsilon \rightarrow 0$ . The standing assumptions are given in A4.1-A4.3. When  $D_{21}^T(\epsilon)$  and  $D_{12}(\epsilon)$  are injective the standard result (see [31]) yields an  $n$ -th order compensator

$$F(\epsilon, s) = -K_{12}(\epsilon)(sI - A + B_2 K_{12}(\epsilon) + K_{21}(\epsilon)C_2)^{-1} K_{21}(\epsilon) \quad (4.20)$$

where

$$K_{12}(\epsilon) := (D_{12}(\epsilon)^T D_{12}(\epsilon))^{-1} (B_2^T P_{12}(\epsilon) + D_{12}^T C_1) \quad (4.21)$$

$$K_{21}(\epsilon) := (P_{21}(\epsilon)C_2^T + B_1 D_{21}^T(\epsilon))(D_{21}(\epsilon)D_{21}(\epsilon)^T)^{-1} \quad (4.22)$$

and  $P_{12}(\epsilon)$ ,  $P_{21}(\epsilon)$  are the unique positive definite stabilizing solutions to (we drop the  $\epsilon$  arguments for brevity)

$$A^T P_{12} + P_{12} A + C_1^T C_1 - K_{12}^T D_{12}^T D_{12} K_{12} = 0 \quad (4.23)$$

$$P_{21} A^T + A P_{21} + B_1 B_1^T - K_{21} D_{21} D_{21}^T K_{21}^T = 0 \quad (4.24)$$

Before proceeding to the main result of this section, let us define the system matrices

$$M_{G_{12}}(s) := \left[ \begin{array}{c|c} -sI + A & B_2 \\ \hline C_1 & D_{12} \end{array} \right] \quad (4.25)$$

$$M_{G_{21}}(s) := \left[ \begin{array}{c|c} -sI + A & B_1 \\ \hline C_2 & D_{21} \end{array} \right] \quad (4.26)$$

Let  $\begin{bmatrix} X_{12_\infty}^{(l)} \\ U_{12_\infty}^{(l)} \end{bmatrix}$  and  $\begin{bmatrix} X_{21_\infty}^{(l)} \\ U_{21_\infty}^{(l)} \end{bmatrix}$  be matrices whose columns form bases for the lower infinite zero eigensubspaces of  $M_{G_{12}}(s)$  and  $M_{G_{21}}^T(s)$  respectively; similarly, let  $\begin{bmatrix} X_{12_\infty}^{(u)} \\ U_{12_\infty}^{(u)} \end{bmatrix}$  and  $\begin{bmatrix} X_{21_\infty}^{(u)} \\ U_{21_\infty}^{(u)} \end{bmatrix}$  be matrices whose columns form bases for the upper infinite zero eigensubspaces of the respective pencils  $M_{G_{12}}(s)$  and  $M_{G_{21}}^T(s)$ .

In addition, it will be assumed that the columns of  $\begin{bmatrix} X_{12_j} \\ U_{12_j} \end{bmatrix}$  and  $\begin{bmatrix} X_{21_j} \\ U_{21_j} \end{bmatrix}$  form bases for the finite  $\mathcal{C}^{0e}$  eigenspaces of  $M_{G_{12}}(s)$  and  $M_{G_{21}}^T(s)$  respectively, corresponding to the matrices  $\Lambda_f = \Lambda_{12_j}$  and  $\Lambda_f = \Lambda_{21_j}$  in (2.22).

To set up the appropriate eigenproblems, we define

$$W_{12}(\epsilon, s) = \left[ \begin{array}{ccc} -sI + A & 0 & B_2 \\ -C_1^T C_1 & -sI - A^T & -C_1^T D_{12}(\epsilon) \\ D_{12}^T(\epsilon) C_1 & B_2^T & D_{12}^T(\epsilon) D_{12}(\epsilon) \end{array} \right] \quad (4.27)$$

$$W_{21}(\epsilon, s) = \begin{bmatrix} -sI + A^T & 0 & C_2^T \\ -B_1 B_1^T & -sI - A & -B_1 D_{21}^T(\epsilon) \\ D_{21}(\epsilon) B_1^T & C_2 & D_{21}(\epsilon) D_{21}^T(\epsilon) \end{bmatrix} \quad (4.28)$$

Now let the columns of  $\begin{bmatrix} X_{12,} \\ \Phi_{12,} \\ U_{12,} \end{bmatrix}$  and  $\begin{bmatrix} X_{21,} \\ \Phi_{21,} \\ U_{21,} \end{bmatrix}$  form bases for the stable subspaces of  $W_{12}(0, s)$  and  $W_{21}(0, s)$  respectively, corresponding to the matrices  $\Lambda_f = \Lambda_{12,}$  and  $\Lambda_f = \Lambda_{21,}$  in (2.22).

Finally define

$$\begin{bmatrix} X_{12} \\ \Phi_{12} \\ U_{12} \end{bmatrix} = \begin{bmatrix} X_{12,} & X_{12,j} & X_{12,\infty}^{(l)} \\ \Phi_{12,} & 0 & 0 \\ U_{12,} & U_{12,j} & U_{12,\infty}^{(l)} \end{bmatrix} \quad (4.29)$$

$$\begin{bmatrix} X_{21} \\ \Phi_{21} \\ U_{21} \end{bmatrix} = \begin{bmatrix} X_{21,} & X_{21,j} & X_{21,\infty}^{(l)} \\ \Phi_{21,} & 0 & 0 \\ U_{21,} & U_{21,j} & U_{21,\infty}^{(l)} \end{bmatrix} \quad (4.30)$$

and note that for all  $\epsilon > 0$  the controller  $F(\epsilon, s)$  has system matrix representation

$$P_F(\epsilon, s) \stackrel{s}{=} \left[ \begin{array}{c|c} \frac{X_{21}^T(\epsilon)(-sI + A)X_{12}(\epsilon) + X_{21}^T(\epsilon)B_2U_{12}(\epsilon) + U_{21}^T(\epsilon)C_2X_{12}(\epsilon)}{-U_{12}(\epsilon)} & U_{21}^T(\epsilon) \\ \hline & 0 \end{array} \right] \quad (4.31)$$

where the columns of  $\Xi_{12}(\epsilon) := \begin{bmatrix} X_{21}(\epsilon) \\ \Lambda_{21}(\epsilon) \\ U_{21}(\epsilon) \end{bmatrix}$  and  $\Xi_{21}(\epsilon) := \begin{bmatrix} X_{12}(\epsilon) \\ \Lambda_{12}(\epsilon) \\ U_{12}(\epsilon) \end{bmatrix}$  form bases for the respective stable eigenspaces of  $W_{21}(\epsilon, s)$  and  $W_{12}(\epsilon, s)$ , i. e.  $\Xi_{12}(\epsilon)$  and  $\Xi_{21}(\epsilon)$  satisfy (2.22) with  $\Lambda_f = \Lambda_{12}(\epsilon)$  and  $\Lambda_f = \Lambda_{21}(\epsilon)$  respectively where  $\Lambda_{12}(\epsilon)$  and  $\Lambda_{21}(\epsilon)$  are asymptotically stable matrices  $\forall \epsilon > 0$ .

The limiting form of the required compensator is given by the following

**Theorem 4.1** *Suppose that  $\exists \epsilon > 0$  for which a solution to the LQG problem (4.19) exists.*

*Then, as  $\epsilon \rightarrow 0$  the following hold:*

*(i) The limiting value of the controller system matrix is*

$$P_F(s) := \left[ \begin{array}{c|c} \frac{X_{21}^T(-sI + A)X_{12} + X_{21}^T B_2 U_{12} + U_{21}^T C_2 X_{12}}{-U_{12}} & \frac{U_{21}^T}{0} \end{array} \right]. \quad (4.32)$$

*(ii) The limiting closed loop transfer function is asymptotically stable and is given by*

$$T_{y_1 u_1}(s) \stackrel{s}{=} \left[ \begin{array}{cc|c} -sI + \Lambda_{12}, & H_{12}^T, (H_{21}, \Lambda_{21}^T, -AH_{21},) & H_{12}^T, B_* \\ 0 & (-sI + \Lambda_{21},)^T & -(X_{21}^T, B_1 + U_{21}^T, D_{21}) \\ \hline C_1 X_{12}, + D_{12} U_{12}, & -C_1 H_{21}, & 0 \end{array} \right] \quad (4.33)$$

where

$$B_* := B_1 - H_{21}^T, (X_{21}^T, B_1 + U_{21}^T, D_{21}) \quad (4.34)$$

and  $H_{12}, H_{21},$  satisfy

$$H_{12}^T := \begin{bmatrix} H_{12}^T \\ H_{12}^T \\ H_{12\infty}^T \end{bmatrix} = \begin{bmatrix} X_{12}, & X_{12}, & X_{12\infty}^{(u)} \end{bmatrix}^{-1} \quad (4.35)$$

$$H_{21}^T := \begin{bmatrix} H_{21}^T \\ H_{21}^T \\ H_{21\infty}^T \end{bmatrix} = \begin{bmatrix} X_{21}, & X_{21}, & X_{21\infty}^{(u)} \end{bmatrix}^{-1}. \quad (4.36)$$

(iii) The limiting cost (4.19) is given by

$$J_* := \lim_{\epsilon \rightarrow 0} J(\epsilon) = \text{Tr}(\Pi_*) \quad (4.37)$$

where

$$\Pi_* = P_{c*} B_* B_*^T + C_1^T C_1 P_{f*} \quad (4.38)$$

$$P_{c*} = \Phi_{12} H_{12}^T \quad (4.39)$$

$$P_{f*} = H_{21} \Phi_{21}^T \quad (4.40)$$

(iv) If the system matrix  $P_F(s)$  is well-posed, then  $\lim_{\epsilon \rightarrow 0} F(\epsilon, s)$  exists and is given

by

$$F(s) \stackrel{s}{=} P_F(s) \quad (4.41)$$

and this value of  $F(s)$  achieves the limiting transfer function  $T_{y_1 u_1}(s)$  and cost  $J_*$ .

*Proof:* It is easy to show that for  $\epsilon > 0$  the stochastic regulator is given by

$$F(\epsilon, s) \stackrel{s}{=} P_F(\epsilon, s) \quad (4.42)$$

The fact that  $P_F(\epsilon, s) \rightarrow P_F(s)$  follows from Theorem 3.1 and the analyticity of these matrices in  $\epsilon$ .

For part (ii) assume that  $\epsilon^*$  is small enough so that those stable eigenvalues of  $W_{12}(\epsilon, s)$  which approach locations at infinity, the finite  $j\omega$ -axis and the open LHP

are all in disjoint groups on  $[0, \epsilon^*]$ . For  $\epsilon > 0$  we know that the closed loop transfer function has Rosenbrock system matrix representation

$$P_{T_{y_1 u_1}}(\epsilon, s) = \left[ \begin{array}{cc|c} -sI + A - B_2 K_{12} & -B_2 K_{12} & B_1 \\ 0 & -sI + A - K_{21} C_2 & K_{21} D_{21} - B_1 \\ \hline C_1 - D_{12} K_{12} & -D_{12} K_{12} & 0 \end{array} \right] \quad (4.43)$$

(4.44)

where  $K_{12}(\epsilon)$ ,  $K_{21}(\epsilon)$  are given in (4.21,4.22). Perform the following equivalence transformations:

column 2 = column 2 - column 1

column 1 = column 1  $\times X_{12}$

row 2 =  $X_{21}^T \times$  row 2.

and use the eigenvector relationships for  $W_{12}(\epsilon, s)$  and  $W_{21}(\epsilon, s)$  in (2.22) to obtain the descriptor form

$$P_{T_{y_1 u_1}}(\epsilon, s) = \left[ \begin{array}{cc|c} X_{12}(\epsilon)(-sI + \Lambda_{12}(\epsilon)) & sI - A & B_1 \\ 0 & (-sI + \Lambda_{21}(\epsilon))^T X_{21}^T(\epsilon) & -(U_{21}^T(\epsilon)D_{21}(\epsilon) + X_{21}^T(\epsilon)B_1) \\ \hline C_1 X_{12}(\epsilon) + D_{12}(\epsilon)U_{12}(\epsilon) & -C_1 & 0 \end{array} \right] \quad (4.45)$$

For  $\epsilon$  small enough we can partition  $\Xi_{12}(\epsilon)$  and  $\Xi_{21}(\epsilon)$  as follows:

$$\Xi_{12}(\epsilon) = \left[ \begin{array}{c|c|c} X_{12_i}(\epsilon) & X_{12_j}(\epsilon) & X_{12_\infty}(\epsilon) \\ \Phi_{12_i}(\epsilon) & \Phi_{12_j}(\epsilon) & \Phi_{12_\infty}(\epsilon) \\ U_{12_i}(\epsilon) & U_{12_j}(\epsilon) & U_{12_\infty}(\epsilon) \end{array} \right] \quad (4.46)$$



$$\Xi_{21}(\epsilon) = \left[ \begin{array}{c|c|c} X_{21_s}(\epsilon) & X_{21_j}(\epsilon) & X_{21_\infty}(\epsilon) \\ \hline \Phi_{21_s}(\epsilon) & \Phi_{21_j}(\epsilon) & \Phi_{21_\infty}(\epsilon) \\ \hline U_{21_s}(\epsilon) & U_{21_j}(\epsilon) & U_{21_\infty}(\epsilon) \end{array} \right] \quad (4.47)$$

where

$$\lim_{\epsilon \rightarrow 0} \left[ \begin{array}{c|c|c} X_{12_s}(\epsilon) & X_{12_j}(\epsilon) & X_{12_\infty}(\epsilon) \\ \hline \Phi_{12_s}(\epsilon) & \Phi_{12_j}(\epsilon) & \Phi_{12_\infty}(\epsilon) \\ \hline U_{12_s}(\epsilon) & U_{12_j}(\epsilon) & U_{12_\infty}(\epsilon) \end{array} \right] = \left[ \begin{array}{c|c|c} X_{12_s} & X_{12_j} & X_{12_\infty}^{(l)} \\ \hline \Phi_{12_s} & \Phi_{12_j} & 0 \\ \hline U_{12_s} & U_{12_j} & U_{12_\infty}^{(l)} \end{array} \right] \quad (4.48)$$

and

$$\lim_{\epsilon \rightarrow 0} \left[ \begin{array}{c|c|c} X_{21_s}(\epsilon) & X_{21_j}(\epsilon) & X_{21_\infty}(\epsilon) \\ \hline \Phi_{21_s}(\epsilon) & \Phi_{21_j}(\epsilon) & \Phi_{21_\infty}(\epsilon) \\ \hline U_{21_s}(\epsilon) & U_{21_j}(\epsilon) & U_{21_\infty}(\epsilon) \end{array} \right] = \left[ \begin{array}{c|c|c} X_{21_s} & X_{21_j} & X_{21_\infty}^{(l)} \\ \hline \Phi_{21_s} & \Phi_{21_j} & 0 \\ \hline U_{21_s} & U_{21_j} & U_{21_\infty}^{(l)} \end{array} \right] \quad (4.49)$$

Associated with those zeros of interest which remain finite as  $\epsilon \rightarrow 0$  are the matrices  $\Lambda_{12_s}(\epsilon)$ ,  $\Lambda_{12_j}(\epsilon)$  whose eigenvalues correspond to the eigenvalues of  $W(\epsilon, s)$  on each subspace pertaining to the closed right half plane; similarly,  $\Lambda_{12_s}(\epsilon)$ ,  $\Lambda_{12_j}(\epsilon)$  are matrices whose eigenvalues correspond to those of  $W_{21}(\epsilon, s)$  on the pertinent subspaces. Moreover, we have that

$$\lim_{\epsilon \rightarrow 0} \Lambda_{12_s}(\epsilon) = \Lambda_{12_s}, \quad \lim_{\epsilon \rightarrow 0} \Lambda_{12_j}(\epsilon) = \Lambda_{12_j}, \quad (4.50)$$

and

$$\lim_{\epsilon \rightarrow 0} \Lambda_{21_s}(\epsilon) = \Lambda_{21_s}, \quad \lim_{\epsilon \rightarrow 0} \Lambda_{21_j}(\epsilon) = \Lambda_{21_j}, \quad (4.51)$$

Substituting in (4.43) and using the equations which define each eigensubspace, we obtain  $P_{T_{y_1 u_1}}(\epsilon, s)$  which is listed in Table 4.1 on page 76. Now multiply row 1 on the left by  $H_{12}^T$  and column 4 on the right by  $H_{21}$  defined in the theorem and

take the limit as  $\epsilon \rightarrow 0$  to obtain  $P_{T_{y_1 u_1}}(s)$  which is the second entry of Table 4.1. Here we have used (4.35, 4.36) and the following facts (Lemma 2.6):

$$\begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} X_{12\infty} \\ U_{12\infty} \end{bmatrix} = \begin{bmatrix} X_{12\infty}^{(u)} \\ 0 \end{bmatrix} \quad (4.52)$$

$$\begin{bmatrix} A^T & C_{21}^T \\ B_1^T & D_{21} \end{bmatrix} \begin{bmatrix} X_{21\infty} \\ U_{21\infty} \end{bmatrix} = \begin{bmatrix} X_{21\infty}^{(u)} \\ 0 \end{bmatrix} \quad (4.53)$$

Clearly the (1,1) block in  $P_{T_{y_1 u_1}}(s)$  is regular  $\Rightarrow$  by Lemma 4.2 that  $T_{y_1 u_1}(s) := \lim_{\epsilon \rightarrow 0} T_{y_1 u_1}(\epsilon, s)$  exists. Observe that this also follows from the fact that  $\forall \epsilon > 0$   $T_{y_1 u_1}(\epsilon, s)$  is an  $RH^2$  function with monotonically decreasing norm. It is readily ascertained that the  $C^{0e}$  blocks in  $P_{T_{y_1 u_1}}(s)$  are either unobservable or uncontrollable. This gives

$$T_{y_1 u_1}(s) \stackrel{s}{=} \left[ \begin{array}{cc|c} -sI + \Lambda_{12}, & H_{12}^T (sI - A) H_{21}, & H_{12}^T B_1 \\ 0 & (-sI + \Lambda_{21})^T & -(U_{21}^T D_{21} + X_{21}^T B_1) \\ \hline C_1 X_{12}, & -C_1 H_{21}, & 0 \end{array} \right] \quad (4.54)$$

Now subtract  $(H_{12}, H_{21}^T)$  times row 2 from row 1 to obtain the result.

To prove (iii) it suffices to note that the limit exists by Lemma 3.4 and to compute the cost directly from (4.33). The reader is referred to Appendix B for the details.

Finally for (iv) note that the existence of the limit in (4.41) is guaranteed by Lemma 4.2. The rest follows from the fact that

$$T_{y_1 u_1}(s) = \mathcal{F}_l(G(s), F(s)) \quad (4.55)$$

where  $G(s)$  is the block transfer function with state-space representation (4.1). The details are worked out in Appendix C.

□

*Comment:* Observe that, as expected, the  $j\omega$ -axis zeros are cancelled in the transfer function  $T_{y_1 u_1}$ ; hence, the limiting feedback control law is not internally stabilizing when there are (finite)  $j\omega$ -axis zeros.

*Comment:* (i) The structure in (4.32) automatically compensates for rank deficiencies in  $D_{21}(0)$  and  $D_{12}(0)$ . Under the appropriate conditions, (4.32) therefore incorporates the usual minimal order observer and/or dual observer structures [17, 41]. This issue will shortly be discussed in more detail.

(ii) Since  $\text{rank}(X_{21}^T X_{12}) \leq n$  it follows that the compensator  $F(s)$  may be improper; since the rank deficiency in  $X_{21}, X_{12}$  is due solely to the existence of infinite zeros in  $G_{12}, G_{21}$  this confirms the fact that  $F(s)$  can only be improper if  $D_{12}$  and/or  $D_{21}^T$  are column rank deficient (see example in Section 4.4). The compensator order  $\delta$  satisfies

$$\delta \leq \min\{\text{rank}[X_{21}, X_{21\infty}], \text{rank}[X_{12}, X_{12\infty}]\} \leq n$$

We complete this section by showing that (4.32) embodies a minimal order observer structure [31, 41] under the following conditions [17]:

**A4.4**  $D_{21}(0) = 0$ .

**A4.5**  $D_{12}(0)$  is injective.

**A4.6**  $C_2 B_1$  is surjective.

We state the result as a corollary to Theorem 4.1.

**Corollary 4.2** Suppose that assumptions A4.4-4.6 hold. Then the limiting controller exists and has  $(n - m_2)$ -th order state-space realization

$$F(s) \stackrel{s}{=} \left[ \begin{array}{c|c} -sI + X^\dagger A_k (I - X_{21,\perp}^\perp (C_2 X_{21,\perp}^\perp)^{-1} C_2) X_{21,\perp} & -X_{21,\perp}^\dagger A_k X_{21,\perp}^\perp (C_2 X_{21,\perp}^\perp)^{-1} \\ \hline K_{12} (I - X_{21,\perp}^\perp (C_2 X_{21,\perp}^\perp)^{-1} C_2) X_{21,\perp} & -K_{12} X_{21,\perp}^\perp (C_2 X_{21,\perp}^\perp)^{-1} \end{array} \right] \quad (4.56)$$

where the columns of  $X_{21,\perp}^\perp$  form a basis for  $\text{Im}^\perp X_{21,\perp}$  and

$$X_{21,\perp}^\dagger := (X_{21,\perp}^T X_{21,\perp})^{-1} X_{21,\perp}^T \quad (4.57)$$

$$A_k := A - B_2 K_{12} \quad (4.58)$$

$$K_{12} = -U_{12} X_{12}^{-1} \quad (4.59)$$

*Proof:* Assumption A6 implies that all the infinite zeros of  $C_2(sI - A)^{-1}B_1 + D_{21}(0)$  are of order 1. By the usual recursions, we see that  $M_{G_{21}}^T(s)$  has  $m_2$  chains at infinity of length 2; in fact the columns of the matrix  $\begin{bmatrix} 0 & C_2^T \\ I_{m_2} & 0 \end{bmatrix}$  form a minimal basis for the infinite frequency eigenspace of  $M_{G_{21}}^T(s)$ . Since  $W_{12}(0, s)$  has an  $n$ -dimensional stable eigenspace we obtain from Theorem 4.1

$$F(s) = \left[ \begin{array}{c|c} X_{21,\perp}^T (-sI + A_k) X_{12,\perp} + U_{21,\perp}^T C_2 X_{12,\perp} & U_{21,\perp}^T \\ \hline C_2 X_{12,\perp} & I_{m_2} \\ -U_{12} & 0 \end{array} \right] \quad (4.60)$$

If  $X_{12,\perp}$  is of rank  $n$  then there exists an  $n \times n$  matrix  $T$  such that  $[X_{21,\perp}^T, X_{21,\perp}^\perp] = X_{12,\perp}^T T$ , where  $X_{21,\perp}^\perp \in \mathbb{R}^{n \times m_2}$ ,  $\text{Im} X_{21,\perp}^\perp = \text{Im}^\perp X_{21,\perp}$ . Multiply the first column of (4.60) by  $T$  and partition to get

$$\left[ \begin{array}{cc|c} X_{21,\perp}^T (-sI + A_k) X_{21,\perp} + U_{21,\perp}^T C_2 X_{21,\perp} & X_{21,\perp}^T A X_{21,\perp}^\perp + U_{21,\perp}^T C_2 X_{21,\perp}^\perp & U_{21,\perp}^T \\ \hline C_2 X_{21,\perp} & C_2 X_{21,\perp}^\perp & I_{m_2} \\ K_{12} X_{21,\perp} & K_{12} X_{21,\perp}^\perp & 0 \end{array} \right] \quad (4.61)$$

Subtract  $U_{21}^T \times$  row 2 from row 1 to get

$$\left[ \begin{array}{cc|c} X_{21}^T(-sI + A_k)X_{21} & X_{21}^T A X_{21}^\perp & U_{21}^T \\ C_2 X_{21} & C_2 X_{21}^\perp & I_{m_2} \\ \hline K_{12} X_{21} & K_{12} X_{21}^\perp & 0 \end{array} \right] \quad (4.62)$$

Now note first of all, that  $X_{21}$ , is of column dimension  $n - m_2$ . Moreover, by Lemma 3.8, the existence of a solution  $\Rightarrow [X_{21}, C_2^T]$  is nonsingular  $\Rightarrow C_2 X_{21}^\perp$  is also a nonsingular matrix. Thus (4.60) has an  $(n - m_2)$ -th order non-strictly proper state-space realization (4.56).  $\square$

*Comment:* By a similar argument, it can be shown that  $F(s)$  has a reduced order dual-observer structure [41] for  $D_{21}(0)$  surjective,  $D_{12}(0)$  column rank deficient and  $C_1 B_2$  injective.

*Comment:* The assumption on  $D_{12}(0)$  is simplified for ease of exposition; however, the above arguments follow through when  $D_{12}(0)$  does not satisfy A4 but is row rank deficient.

$$P_{T_{y_1 u_1}}(\epsilon, s) = \begin{bmatrix} X_{12_s}(\epsilon)(-sI + \Lambda_{12_s}(\epsilon)) & X_{12_j}(\epsilon)(-sI + \Lambda_{12_j}(\epsilon)) & (-sI + A)X_{12_\infty}(\epsilon) + B_2 U_{12_\infty}(\epsilon) & sI - A & B_1 \\ 0 & 0 & 0 & (-sI + \Lambda_{21_s}(\epsilon))^T X_{21_s}^T(\epsilon) & -(U_{21_s}^T(\epsilon)D_{21} + X_{21_s}^T(\epsilon)B_1) \\ 0 & 0 & 0 & (-sI + \Lambda_{21_j}(\epsilon))^T X_{21_j}^T(\epsilon) & -(U_{21_j}^T(\epsilon)D_{21} + X_{21_j}^T(\epsilon)B_1) \\ 0 & 0 & 0 & X_{21_\infty}^T(\epsilon)(-sI + A) + U_{21_\infty}^T C_{21}^T & -(U_{21_\infty}^T(\epsilon)D_{21} + X_{21_\infty}^T(\epsilon)B_1) \\ C_1 X_{12_s}(\epsilon) + D_{12} U_{12_s}(\epsilon) & C_1 X_{12_j}(\epsilon) + D_{12} U_{12_j}(\epsilon) & (C_1 X_{12_\infty}(\epsilon) + D_{12} U_{12_\infty}(\epsilon)) & -C_1 & 0 \end{bmatrix}$$

(a)

$$P_{T_{y_1 u_1}}(s) := \lim_{\epsilon \rightarrow 0} P_{T_{y_1 u_1}}(\epsilon, s) =$$

$$\begin{bmatrix} -sI + \Lambda_{12_s} & 0 & 0 & * & * & H_{12_s}^T B_1 \\ 0 & -sI + \Lambda_{12_j} & 0 & * & * & H_{12_j}^T B_1 \\ 0 & 0 & -sH_{12_\infty}^T X_{12_\infty} + I & * & * & H_{12_\infty}^T B_1 \\ 0 & 0 & 0 & (-sI + \Lambda_{21_s})^T & 0 & -(U_{21_s}^T D_{21} + X_{21_s}^T B_1) \\ 0 & 0 & 0 & 0 & (-sI + \Lambda_{21_j})^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 X_{12_s} + D_{12} U_{12_s} & 0 & 0 & -C_1 H_{21_s}^T & -sX_{21_\infty}^T H_{21_\infty} + I & 0 \\ C_1 X_{12_j} + D_{12} U_{12_j} & 0 & 0 & -C_1 H_{21_j}^T & -C_1 H_{21_\infty}^T & 0 \end{bmatrix}$$

(b)

Table 4.1: System matrices for  $T_{y_1 u_1}(s)$

## 4.4 Computational Aspects and an Example

We will now discuss the issue of the computation of the subspaces described in Theorems 3.1, 4.1. The finite subspaces of  $W_{21}(\epsilon, s)$ ,  $W_{12}(\epsilon, s)$  may, of course, be determined by application of the QZ algorithm [10]. Van Dooren [11], however, has pointed out that this approach may be rendered numerically unreliable by the infinite frequency structure of these pencils. Our computational procedures are therefore based on the "pencil algorithms" discussed in [11].

### Algorithm 1: Finite subspace calculation

**Input:** State space parameters of  $G(s) \triangleq \left[ \begin{array}{c|c} -sI + A & B \\ \hline C & D \end{array} \right]$ .

**Step 1** Form the pencil  $W(s) = \left[ \begin{array}{cc|c} -sI + A & 0 & B \\ -C^T C & -sI - A^T & -C^T D \\ \hline D^T C & B^T & D^T D \end{array} \right]$ .

**Step 2** Use the dual form of Algorithm 3.6 [11] to obtain unitary matrices  $S_1, T_1$  such that

$$S_1 W(s) T_1 = \left[ \begin{array}{c|c} -sE_\infty + A_\infty & 0 \\ \hline * & -sE_f + A_f \end{array} \right] \quad (4.63)$$

where the (2,2) block is regular and contains the finite frequency structure of  $W(s)$ .

**Step 3** Using the QZ algorithm or block Schur procedures [6] determine unitary matrices  $S_2, T_2$  such that

$$S_2 (-sE_f + A_f) T_2 = \left[ \begin{array}{c|c} -sE_+ + A_+ & 0 \\ \hline * & -sE_- + A_- \end{array} \right] \quad (4.64)$$

where the (2,2) block contains all the LHP zeros of  $-sE_f + A_f$  and hence  $W(s)$ .

**Step 4** Let  $S = \left[ \begin{array}{c|c} I & 0 \\ \hline 0 & S_2 \end{array} \right] S_1$ ,  $T = T_1 \left[ \begin{array}{c|c} I & 0 \\ \hline 0 & T_2 \end{array} \right]$ . Then, with the obvious partitioning,

$$\left[ \begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array} \right] W(s) \left[ \begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array} \right] = \left[ \begin{array}{c|c} * & 0 \\ \hline * & -sE_- + A_- \end{array} \right] \quad (4.65)$$

where the (2,2) on the right hand side again contains all the LHP zeros of  $W(s)$ .

The required basis vectors are then the columns of  $\begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}$ .

**Output:**  $\begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}$  whose columns form a minimal basis for the stable eigenspace of  $W(s)$ .

### Algorithm 2: Infinite frequency subspace

**Input:** State space parameters of  $G(s) \stackrel{s}{=} M(s)$  where  $M(s) = \left[ \begin{array}{c|c} -sI + A & B \\ \hline C & D \end{array} \right]$ .

**Step 1** Use algorithm 4.1 of [11] to obtain unitary matrices  $S, T$  such that (partitioning  $S, T$  accordingly)

$$\left[ \begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array} \right] M(s) \left[ \begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array} \right] = \left[ \begin{array}{c|c} * & 0 \\ \hline * & -sE_\infty + A_\infty \end{array} \right] \quad (4.66)$$

where the (2,2) block contains the infinite frequency structure of  $M(s)$ .

**Step 2** The columns of  $\Xi_\infty = \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}$  form a minimal basis for the required eigenspace. Apply the SVD [10] to obtain an orthogonal matrix  $V_1$  which spans the right null-space of  $\begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}$ .



Then the columns of  $\hat{\Xi}_{\infty}^{(l)} = \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix} V_1$  form a minimal basis for the lower infinite eigenspace of  $M(s)$ .

**Step 3** Again use the SVD to determine a unitary matrix  $V_2$  whose columns form a minimal basis for  $\text{Ker } {}^{\perp}T_{12}$ . Then the columns of the matrix  $\Xi_{\infty}^{(u)} \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix} V_2$  form a minimal basis for the upper infinite frequency eigensubspace of  $M(s)$ .

**Output:**  $\begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}$ ,  $\Xi_{\infty}^{(l)}$  and  $\Xi_{\infty}^{(u)}$ .

*Comment:* Justification for Steps 2 and 3 above follows directly from Lemma 2.8.

The theory developed above will now be applied to

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -b & 1 & 0 \\ 0 & -a & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2 \quad (4.67)$$

$$y_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2 \quad (4.68)$$

$$y_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 & 0 & \epsilon \end{bmatrix} u_1 \quad (4.69)$$

where  $a > 0$ ,  $b > 0$  and  $u_1 = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$  is unit intensity white noise.

Forming the appropriate pencils we find that  $W_{12}(0, s)$  has LHP zeros at  $s = -1, -b, -a$  with

$$X_{12_s} = \begin{bmatrix} -1 & 1 & 1 \\ 1-b & 0 & b-a \\ (a-1)(1-b) & 0 & 0 \end{bmatrix}, \quad \Phi_{12_s} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (a-1)(1-b) & 0 & 0 \end{bmatrix}$$

and  $U_{12_s} = [(1-a)(1-b) \ 0 \ 0]$ . Also  $W_{21}(0, s)$  has only one LHP eigenvalue (at  $s = -1$ ).

The required matrices here are

$$X_{21_s} = \begin{bmatrix} a-1 \\ 1 \\ -1 \end{bmatrix}, \quad \Phi_{21_s} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad U_{21_s} = [(1-a)(1-b)]$$

for the infinite frequency eigenspace

$$X_{21_\infty} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad U_{21_\infty} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Substitution in (4.32) yields

$$F(s) \stackrel{s}{=} \left[ \begin{array}{ccc|c} (s+1)(2(a+b) - ab - 3) & (s+b)(1-a) & (s+a)(1-b) & 0 \\ -1 & 1 & 1 & 1 \\ s+1 & -(s+b) & -(s+a) & 0 \\ \hline (a-1)(1-b) & 0 & 0 & 0 \end{array} \right] \quad (4.70)$$

which is well posed for  $a \neq 1, b \neq 1, a \neq b$  with corresponding transfer function

$$F(s) = -\frac{(s+a)(s+b)}{s+2}. \quad (4.71)$$

There is a nice interpretation of the structure derived. Note first that the system in question has the cascade structure shown in Fig 4.1a; for  $\epsilon = 0$  the output is

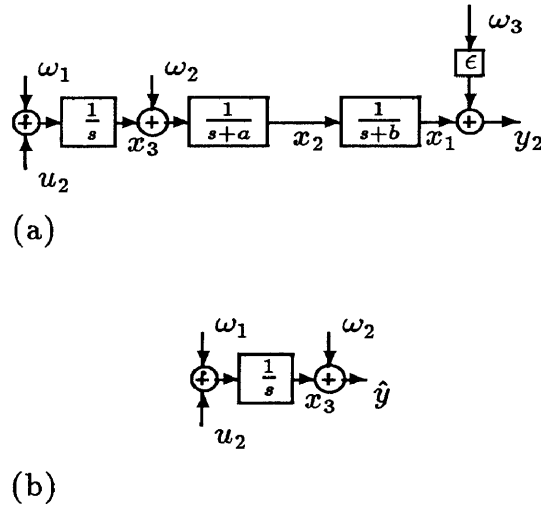


Figure 4.1: Block Diagrams for LQG Example

noise free and can therefore be (theoretically) differentiated via the transfer function  $(s + a)(s + b)$  to obtain the precise value of  $\hat{y} = \begin{pmatrix} x_3 \\ \omega_2 \end{pmatrix}$ . Since the noise intensities for the remaining subsystem (see Fig 4.1b) are nonsingular we can now consider the reduced (regular) LQG problem of determining  $u_2(t)$  to minimize  $E(x_3^2 + u_2^2)$ . The dynamic constraint equations are  $\dot{x}_3 = u_2 + \omega_1$ , with measured variable  $\hat{y}$ . The stabilizing solution to this problem is  $\hat{F}(s) = -\frac{1}{s+2}$ . Now note that  $F(s) = \hat{F}(s)(s + a)(s + b)$ , implying that the procedure generated by Theorems 3.1, 4.1 automatically assumes this particular cancellation structure.

Finally, observe that while the transfer function (4.71) exists  $\forall a, b$ , the system matrix (4.70) is not i/o well-defined for  $a = b = 1$ . In this case the problem arises due to the fact that  $W_{12}(0, s)$  assumes a third order pole at  $s = -1$ ; recalculation of the appropriate eigenspace for these parameter values yields, as expected,  $F(s) = \frac{-(s+1)^2}{s+2}$ .

## 4.5 Summary and Conclusion

In this chapter we have studied a general form of the cheap LQG problem. The main result is stated in Theorem 4.1 which describes the system matrices for the limiting controller, closed loop transfer function,  $T_{y_1 u_1}(s)$ , and the associated cost. In particular, it has been shown that the limiting controller system matrix is as given in (4.32) and that the limiting controller exists if this system matrix is well-posed.

Theorem 4.1 also shows that despite the fact that the limiting controller may not exist, the limiting closed loop system always exist and has transfer function representation (4.33). The limiting controller always exists when the  $\mathcal{C}^{0e}$  zeros are finite; it may also exist when the infinite zeros are of order at most 1. In the latter case, the controller has a reduced order realization, the reduction being equal to the number of infinite zeros. This is in agreement with already existing results [41]. However, it is important to note that the implied design procedure is more straightforward than that used in practice in that it makes no distinction between observer and dual-observer structures of any order. This suggests the possibility of obtaining realizable solutions to the problem when, say, both  $D_{12}$  and  $D_{21}$  are rank deficient. Moreover, the descriptor framework used allows for the determination of improper as well as proper solutions.

We have also considered the effect of  $j\omega$ -axis zeros in this work, despite the fact that previous results indicate that the optimal solution would incorporate  $j\omega$ -axis poles to cancel the offending zeros. However, although such solutions are not of

any immediate practical significance, they do provide a suitable starting point for suboptimal strategies employing  $H^2$  techniques [14] with more desirable frequency domain properties.

## Chapter 5

### The Cheap $H^\infty$ Problem

#### 5.1 Introduction

In the previous chapter, we analyzed the solution to the cheap LQG problem. In what follows, we continue in the same vein by studying the equivalent  $H^\infty$  problem.

Again we consider the plant  $G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{11}(s) \end{bmatrix}$  with Rosenbrock system matrix representation (RSM) [45]

$$G(s) \stackrel{\Delta}{=} \left[ \begin{array}{c|cc} -sI + A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (5.1)$$

where  $A \in \mathfrak{R}^{n \times n}$ ,  $B_i \in \mathfrak{R}^{n \times r_i}$ ,  $C_i \in \mathfrak{R}^{n \times m_i}$ ,  $i \in \underline{2}$ <sup>1</sup>. It is assumed that  $\{C_2, A\}$  is detectable,  $\{A, B_2\}$  is stabilizable,  $r_1 \geq m_2$  and  $m_1 \geq r_2$ . In the usual  $H^\infty$  control setting,  $G(s)$  is composed of a plant augmented with a set of prescribed weighting functions (see, for example, Fig. 1.1).

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<sup>1</sup>We define  $\underline{n} := \{1, 2, \dots, n\}$

The optimal  $H^\infty$  control problem requires that we determine a controller  $K(s)$  which stabilizes (5.1) and minimizes  $\|T_{y_1 u_1}\|_\infty$  where

$$T_{y_1 u_1}(s) := \mathcal{F}_l(G(s), K(s)) \quad (5.2)$$

and  $\mathcal{F}_l(G(s), K(s))$  denotes the lower linear fractional transformation of  $G(s)$  and  $K(s)$ ,

$$\mathcal{F}_l(G(s), K(s)) := G_{11}(s) + G_{12}K(I - G_{22}K)^{-1}G_{21}(s). \quad (5.3)$$

$T_{y_1 u_1}$  is the closed loop transfer function from  $u_1$  to  $y_1$  (see Fig. 5.1).

For the most part we will be interested in the suboptimal case where it is required to find a controller  $K(s)$  which stabilizes  $G(s)$  and which, for specified  $\gamma > 0$ , results in

$$\|T_{y_1 u_1}\|_\infty < \gamma. \quad (5.4)$$

We assume, without loss of generality, that  $\gamma = 1$ . If this is not the case, the required solution may be determined by appropriately scaling the problem.

In the literature, it is often assumed that  $G_{12}(s)$  and  $G_{21}(s)$  have no  $\mathcal{C}^{0e}$  zeros. If this assumption does not hold, then no stabilizing solutions to the underlying Riccati equations exist; in fact, if there are any infinite zeros, the equation itself is not well-defined. In view of this, and in keeping with our definition in Chapter 1, we say that the  $H^\infty$  problem is singular if  $G_{12}(s)$  and/or  $G_{21}(s)$  have zeros on  $\mathcal{C}^{0e}$ . Otherwise, the problem will be said to be regular. It is to be noted that the regularity condition is overly restrictive since it eliminates a large variety of very

practical systems from the scope of the  $H^\infty$  theory; the  $H^\infty$  problem for plants with integrator action, for example [47] is a singular problem.

Various techniques exist for solving the regular suboptimal problem (see, for example, [14, 36, 50]) when certain detectability/stabilizability assumptions are satisfied. It has been shown [14, 36] that the controller  $K(s)$  may be parameterized by a fixed central controller  $K_0(s)$  and an arbitrary stable contraction  $Q(s)$  (again see Fig. 5.1). No direct solution has as yet been obtained for the optimal  $H^\infty$  control problem, the latter being solved by iteration on the bound  $\gamma$  in the suboptimal problem.

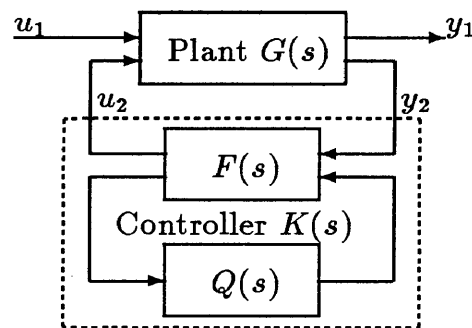


Figure 5.1: Setup for the Standard  $H^\infty$  Problem

In what follows, we approach the singular  $H^\infty$  problem just as we tackled the LQG problem in Chapter 4, i. e. by perturbing it into a regular one by means of a scalar parameter  $\epsilon > 0$ . Unlike previously published results, however (see [67], for example), we consider the more general case of measurement feedback (5.1). More importantly, we also solve the problem of determining the limiting form of



the required controller as  $\epsilon \rightarrow 0$ . We call this new problem a cheap  $H^\infty$  problem in keeping with the equivalent LQ problem.

The analysis herein hinges on the representation of the two Riccati equations as generalized eigenvalue problems. A preliminary step in this direction was made in [48, 19] where the Riccati equations are replaced by the appropriate standard eigenvalue problems. The role of the generalized eigenvalue problems follows directly from the game-theoretic approach taken in [36], for example and, as we shall see, from the standard eigenvalue problem approach in [35] as well. One advantage of the generalized eigen-problem formulation, is that it does not require any precompensation to give the plant  $D_{21}$  and  $D_{12}$  matrices the orthogonal structure necessary for the application of the procedures discussed in [14, 49, 35]. Hence it is expected that the pertinent computations can be reliably performed even when the matrices  $D_{12}$  and  $D_{21}$  are “nearly rank deficient”.

Due to the complexity of the pertinent equations it is still assumed, without loss of generality, that

$$\text{A5.1 } D_{11} = 0, \text{ and}$$

$$\text{A5.2 } D_{22} = 0.$$

As noted previously Safonov *et al.* [49] and Limebeer *et al.* [35] discuss how these restrictions may be easily overcome using loop-shifting procedures.

## 5.2 A New Setting for the Regular Problem

In this section we reformulate the solution to the regular  $H^\infty$  problem in terms of two generalized eigenproblems. This approach differs somewhat from that of [14, 19, 35] and allows for a compact description of the solution when  $D_{12}$  and  $D_{21}$  are not parts of orthogonal matrices, as required in these works. In particular, the procedure eliminates the relevant loop shifting steps in [50] and, more importantly, sets the foundation upon which the analyses performed in later sections of this work are based.

We first consider the case where the states are available for measurement, i. e.  $C_2 = I_n$  and  $D_{21} = 0$ . Define the matrix pencil

$$W_{12}(s) := \begin{bmatrix} -sI + A & B_1 B_1^T & B_2 \\ -C_1^T C_1 & -sI - A^T & -C_1^T D_{12} \\ D_{12}^T C_1 & B_2^T & D_{12}^T D_{12} \end{bmatrix}. \quad (5.5)$$

**Lemma 5.1** *Assume that  $C_2 = I$  and  $D_{12} = 0$ . Let the columns of  $\Xi_{12} = \begin{bmatrix} X_{12} \\ \Phi_{12} \\ U_{12} \end{bmatrix}$  form a basis for the stable eigenspace of  $W_{12}(s)$ . Then there exists a controller which stabilizes  $G(s)$  and makes  $\|T_{y_1 u_1}\|_\infty < 1$  iff*

(i)  $\text{rank } X_{12} = n$ .

(ii)  $\Phi_{12} X_{12}^{-1} \geq 0$ .

If these conditions hold, then one possible solution is the state feedback  $u = Kx$  where

$$K = U_{12}X_{12}^{-1} \quad (5.6)$$

*Proof:* First apply the input transformation

$$u_2 = (D_{12}^T D_{12})^{-\frac{1}{2}} \hat{u}_2 \quad (5.7)$$

and consider the resulting system described by

$$\hat{G}(s) \stackrel{s}{=} \left[ \begin{array}{c|cc} -sI + A & B_1 & \hat{B}_2 \\ \hline C_1 & 0 & \hat{D}_{12} \\ I_n & 0 & 0 \end{array} \right] \quad (5.8)$$

where

$$\hat{B}_2 := B_2(D_{12}^T D_{12})^{-\frac{1}{2}} \quad (5.9)$$

$$\hat{D}_{12} := D_{12}(D_{12}^T D_{12})^{-\frac{1}{2}} \quad (5.10)$$

Now note that  $\hat{D}_{12}$  is part of an orthogonal matrix and apply the loop-shifting transformations of [50] to show that the pertinent Riccati equation is

$$\left. \begin{aligned} (A - \hat{B}_2 \hat{D}_{12}^T C_1)^T P + P(A - \hat{B}_2 \hat{D}_{12}^T C_1) + P(B_1 B_1^T - \hat{B}_2 \hat{B}_2^T) P \\ + C_1^T (I - \hat{D}_{12} \hat{D}_{12}^T) C_1 = 0 \end{aligned} \right\} \quad (5.11)$$

It is known that (5.11) has a positive semidefinite stabilizing solution,  $P$ , iff [50]

(i) the matrix

$$H^{(\infty)} = \left[ \begin{array}{cc} A - \hat{B}_2 \hat{D}_{12}^T C_1 & B_1 B_1^T - \hat{B}_2 \hat{B}_2^T \\ -C_1^T (I - \hat{D}_{12} \hat{D}_{12}^T) C_1 & -(A - \hat{B}_2 \hat{D}_{12}^T C_1)^T \end{array} \right] \quad (5.12)$$

$$= \begin{bmatrix} A - B_2(D_{12}^T D_{12})^{-1} D_{12}^T C_1 & B_1 B_1^T - B_2^T (D_{12}^T D_{12})^{-1} B_2 \\ -C_1^T (I - D_{12} (D_{12}^T D_{12})^{-1} D_{12}^T) C_1 & -(A - B_2 (D_{12}^T D_{12})^{-1} D_{12}^T C_1)^T \end{bmatrix}. \quad (5.13)$$

has an  $n$ -dimensional stable eigenspace and,

(ii) The stable eigenspace of  $H^{(\infty)}$  is spanned by the  $2n \times n$  partitioned matrix  $\begin{bmatrix} X \\ \Phi \end{bmatrix}$  where  $X$  is nonsingular.

The solution to (5.11) is then given by  $P = \Phi X^{-1}$ . Now let  $\Lambda$  be an  $n \times n$  strictly Hurwitz matrix satisfying

$$H^{(\infty)} \begin{bmatrix} X \\ \Phi \end{bmatrix} = \begin{bmatrix} X \\ \Phi \end{bmatrix} \Lambda \quad (5.14)$$

then it is easily verified by direct substitution that

$$W_{12\infty}(s) \begin{bmatrix} X \\ \Phi \\ U \end{bmatrix} = \begin{bmatrix} X \\ \Phi \\ 0 \end{bmatrix} (-sI + \Lambda) \quad (5.15)$$

where

$$U := -(D_{12}^T D_{12})^{-1} (D_{12}^T C_1 X + B_2^T \Phi) \quad (5.16)$$

i. e. the matrix  $\begin{bmatrix} X \\ \Phi \\ U \end{bmatrix}$  spans the stable eigenspace of  $W_{12\infty}(s)$ .

The stabilizing feedback for  $\hat{G}(s)$  is

$$\hat{K} = -(\hat{D}_{12}^T C_1 + \hat{B}_2^T P) \quad (5.17)$$

$\Rightarrow$  the actual feedback is

$$K = -(D_{12}^T D_{12})^{-1} (D_{12}^T C_1 + B_2^T P) = U X^{-1}. \quad (5.18)$$

The proof is completed by letting  $X_{12} = X$ ,  $\Phi_{12} = \Phi$ ,  $U_{12} = U$ .  $\square$

*Comment:* The proof of the foregoing lemma also follows from the linear quadratic differential game (LQDG) approach taken in [36, 38]. The objective there is to determine

$$\min_{u_2} \max_{u_1} \{J(u_1, u_2) = \int_0^\infty (y_1^T y_1 - u_1^T u_1) dt\} \quad (5.19)$$

The application of the Euler-Lagrange equations leads to the following result which describes the role of the pencil  $W_{12}(s)$  in the  $H^\infty$  problem:

**Lemma 5.2** *The following hold for the LQDG (5.19). Consider the homogeneous system*

$$W_{12}(s) \begin{pmatrix} x(s) \\ \phi(s) \\ u(s) \end{pmatrix} = - \begin{pmatrix} x(0) \\ \phi(0) \\ 0 \end{pmatrix} \quad (5.20)$$

(i) *For  $u_1$  free, the optimal inputs in the LQDG (5.19) are given by*

$$u_1^*(s) = B_1^T \phi(s) \quad (5.21)$$

$$u_2^*(s) = Kx(s) \quad (5.22)$$

where  $K$  is given in (5.18) and  $x(s)$ ,  $\phi(s)$  determined from the solution of (5.20).

For any  $u_1(t)$ ,  $u_2(t)$  the corresponding cost is given by

$$J(u_1, u_2, x(0)) = x(0)^T P x(0) + \|u - u^*\|_2^2 - \|u_1 - u_1^*\|_2^2. \quad (5.23)$$

, Thus, for  $x(0) = 0$ ,  $J(u_1^*, u_2^*) = 0$  and  $u_1 = u_1^*$  may be considered as the worst case disturbance input.

(ii) [Stability] For  $u_1 = u_1^*$  and given any  $x(0) \in \mathbb{R}^n$ , the corresponding optimal state trajectory is given by  $x(s)$  in (5.20) if there exists a suitable value of  $\phi(0)$  which results in  $\lim_{t \rightarrow \infty} \phi(t) = 0$ . For stability, we also require that  $\lim_{t \rightarrow \infty} x(t) = 0$ ; this

is possible iff  $\text{rank } X = n$ , in which case the trajectory of  $\begin{pmatrix} x(s) \\ \phi(s) \\ u(s) \end{pmatrix}$  always lies in the stable eigensubspace of  $W_{12}(s)$ .

(iii) For  $x(0) = 0$  and  $\forall u_1 \in \mathcal{L}_2$  the corresponding output  $z$  is also in  $\mathcal{L}_2$  and  $J(u_1, u_2^*) \leq 0 \Rightarrow \|T_{y_1 u_1}\|_\infty \leq 1$ . Furthermore,  $A + BK$  is strictly Hurwitz  $\Rightarrow T_{y_1 u_1} \in \mathcal{RH}^\infty$ .

Now define

$$W_{21}(s) := \begin{bmatrix} -sI + A^T & C_1^T C_1 & C_2^T \\ -B_1 B_1^T & -sI - A & -B_1 D_{21}^T \\ D_{21} B_1^T & C_2 & D_{21} D_{21}^T \end{bmatrix} \quad (5.24)$$

The following lemma states the result for the  $H^\infty$  problem with measurement feedback:

**Lemma 5.3** *There exists a compensator  $u(s) = K(s)y(s)$  which stabilizes  $G(s)$  and*

*makes  $\|T_{y_1 u_1}\|_\infty < 1$  iff there exist matrices  $\Xi_{12} := \begin{bmatrix} X_{12} \\ \Phi_{12} \\ U_{12} \end{bmatrix} \in \mathbb{R}^{(2n+r) \times n}$ ,  $\Xi_{21} := \begin{bmatrix} X_{21} \\ \Phi_{21} \\ U_{21} \end{bmatrix} \in \mathbb{R}^{(2n+m) \times n}$  whose columns form bases for the respective stable eigenspace of  $W_{12}(s)$  and  $W_{21}(s)$ , i. e.*

$$W_{12}(s)\Xi_{12} = \Xi_{12}(-sI + \Lambda_{12}) \quad (5.25)$$

and

$$W_{21}(s)\Xi_{21} = \Xi_{21}(-sI + \Lambda_{21}) \quad (5.26)$$

where  $\Lambda_{12}$  and  $\Lambda_{21}$  are strictly Hurwitz matrices, and the following conditions hold:

(i)  $\text{rank } X_{12} = \text{rank } X_{21} = n.$

(ii)  $\Phi_{12}X_{12}^{-1} \geq 0, X_{21}^{-T}\Phi_{21}^T \geq 0.$

(iii)  $\lambda_{\max}(X_{21}^{-T}\Phi_{21}^T\Phi_{12}X_{12}^{-1}) < 1.$

Furthermore, if these conditions hold, all such compensators may be parameterized by  $K(s) = \mathcal{F}_l(F(s), Q(s))$  where

$$F(s) \stackrel{s}{=} M_F(s) := \left[ \begin{array}{c|cc} -sE_k + A_k & B_{k_1} & B_{k_2} \\ \hline -C_{k_1} & 0 & D_{k_1} \\ -C_{k_2} & D_{k_2} & 0 \end{array} \right] \quad (5.27)$$

$$\left. \begin{aligned} E_k &= X_{21}^T X_{12} - \Phi_{21}^T \Phi_{12} \\ B_{k_1} &= -U_{21}^T & B_{k_2} &= X_{21}^T B_2 + \Phi_{21}^T C_1^T D_{12} \\ C_{k_1} &= -U_{12} & C_{k_2} &= C_2 X_{12} + D_{21} B_1^T \Phi_{12} \\ D_{k_1} &= (D_{12}^T D_{12})^{\frac{1}{2}} & D_{k_2} &= (D_{21} D_{21}^T)^{\frac{1}{2}} \\ A_k &= E_k \Lambda_{12} + U_{21}^T C_{k_2} = \Lambda_{21}^T E_k + B_{k_2} U_{12} \end{aligned} \right\} \quad (5.28)$$

and

$$Q(s) = D_{k_1}^{-1} \hat{Q}(s) D_{k_2}^{-1} \quad (5.29)$$

where  $\|\hat{Q}(s)\|_{\infty} \leq 1, \hat{Q}(s) \in RH^{\infty}.$

*Proof:* It suffices to show that the Lemma yields the result in [19], for example. To do this, first apply the static pre- and post- compensators (see Fig. 5.2)  $u_2 = D_{k_1}^{-1} \hat{u}_2$

and  $\hat{y}_2 = D_{k_2}^{-1} y_2.$  The new plant  $\hat{G}(s) \stackrel{s}{=} \left[ \begin{array}{c|cc} -sI + A & B_1 & B_2 D_{k_1}^{-1} \\ \hline C_1 & 0 & \hat{D}_{12} \\ D_{k_2}^{-1} C_2 & \hat{D}_{21} & 0 \end{array} \right],$  where  $\hat{D}_{12}$  and

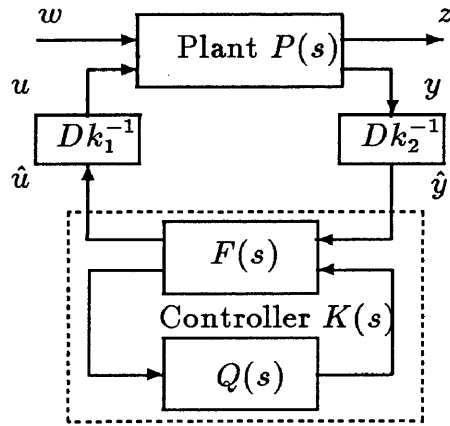


Figure 5.2: Application of Orthogonalizing Compensators to  $G(s)$

$\hat{D}_{21}$  are parts of orthogonal matrices. For this system we form the corresponding

$\hat{W}_{12}(s)$ ,  $\hat{W}_{21}(s)$  along the lines prescribed in (5.5, 5.24) and obtain the matrices

$$\begin{bmatrix} \hat{X}_{12} \\ \hat{\Phi}_{12} \\ \hat{U}_{12} \end{bmatrix} \text{ and } \begin{bmatrix} \hat{X}_{21} \\ \hat{\Phi}_{21} \\ \hat{W}_{21} \end{bmatrix} \text{ whose columns form bases for the respective stable eigenspaces}$$

of  $\hat{W}_{12}(s)$  and  $\hat{W}_{21}(s)$ . By Lemma 5.1 there exist solutions to the usual pair of Riccati

equations in [35], say, iff  $\hat{X}_{12}$ ,  $\hat{X}_{21}$  are nonsingular and  $\hat{\Phi}_{12}\hat{X}_{12}^{-1} \geq 0$ ,  $\hat{X}_{21}^{-T}\hat{\Phi}_{21}^T \geq 0$ .

Hence the controller  $\hat{K}(s)$  for the compensated plant may be determined from [19,

Theorem 5.2].

It is easy to show that 
$$\begin{bmatrix} \hat{X}_{12} \\ \hat{\Phi}_{12} \\ \hat{U}_{12} \end{bmatrix} = \begin{bmatrix} X_{12} \\ \Phi_{12} \\ D_{k_1}^2 U_{12} \end{bmatrix} \text{ and } \begin{bmatrix} \hat{X}_{21} \\ \hat{\Phi}_{21} \\ \hat{U}_{21} \end{bmatrix} = \begin{bmatrix} X_{21} \\ \Phi_{21} \\ D_{k_2}^2 U_{21} \end{bmatrix}. \text{ The rest}$$

follows by noting that  $K(s) = D_{k_1}^{-1}\hat{K}(s)D_{k_2}^{-1}$ . □

*Comment:* If the conditions stated in Theorem 5.3 hold then the solutions to the

usual regulator and filter Riccati equations are given by

$$P = \Phi_{12}X_{12}^{-1} \tag{5.30}$$



$$Q = X_{21}^{-T} \Phi_{21}^T \quad (5.31)$$

respectively.

*Comment:* Following is an alternative expression for  $A_k$  which will prove useful later on:

$$A_k = \Xi_{21}^T \begin{bmatrix} A & B_1 B_1^T & B_2 \\ C_1^T C_1 & A^T & C_1^T D_{12} \\ C_2 & D_{21} B_1^T & 0 \end{bmatrix} \Xi_{12} \quad (5.32)$$

*Comment:* Note that  $Q(s)$  now need not be a contractive operator, although it still must be stable. In view of this fact,  $Q(s)$  will be called an *auxiliary compensator*. When  $Q(s) \equiv 0$ , the corresponding compensator is often referred to, in the literature, as the central controller. We will denote the central controller transfer function by  $K_0(s)$  which is therefore given by

$$K_0(s) \stackrel{s}{=} \left[ \begin{array}{c|c} -sE_k + A_k & B_{k_1} \\ \hline C_{k_1} & 0 \end{array} \right] \quad (5.33)$$

### 5.3 A Regular Embedding for Singular $H^\infty$

#### Problems

We now describe what is essentially a suboptimal design method for the case where  $G_{12}(s)$  and/or  $G_{21}(s)$  have  $\mathcal{C}^{0e}$  zeros. Specifically, we show how to solve the singular problem by determining a solution to a “nearby” regular problem. This therefore represents an extension to the work of [67, 43] in that we treat the more general

case of measurement feedback. As such, we hasten to point out that the procedure is self contained and would, if conditions permit, generate a solution to the singular problem using compensators of at most  $n$ -th order.

Consider the following embedding of the system (5.1):

$$\tilde{G}(\epsilon, s) \stackrel{s}{=} \left[ \begin{array}{c|cc} -sI + A & \tilde{B}_1 & B_2 \\ \hline \tilde{C}_1 & 0 & \tilde{D}_{12} \\ C_2 & \tilde{D}_{21} & 0 \end{array} \right] \quad (5.34)$$

where  $\epsilon > 0$  and

$$\left. \begin{aligned} \tilde{B}_1 &:= \begin{bmatrix} B_1 & 0 \end{bmatrix} \\ \tilde{C}_1 &:= \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \\ \tilde{D}_{12} &:= \begin{bmatrix} D_{12} \\ \epsilon I_{r_2} \end{bmatrix} \\ \tilde{D}_{21} &:= \begin{bmatrix} D_{12} & \epsilon I_{m_2} \end{bmatrix} \end{aligned} \right\} \quad (5.35)$$

Figure 5.3 gives a diagrammatic representation of this embedding.

The new exogenous signals are

$$\tilde{u}_1 = \begin{bmatrix} u_1 \\ u_{1\epsilon} \end{bmatrix} \in \mathfrak{R}^{r_1+m_2} \quad \text{and} \quad \tilde{y}_1 = \begin{bmatrix} y_1 \\ y_{1\epsilon} \end{bmatrix} \in \mathfrak{R}^{m_1+r_2} \quad (5.36)$$

where  $u_{1\epsilon}$  and  $y_{1\epsilon}$  are given by  $y_{1\epsilon} = \epsilon u_2$  and  $y_2 = C_{22}\nu + \epsilon u_{1\epsilon}$ ,  $\nu$  being the state of the system (5.34).

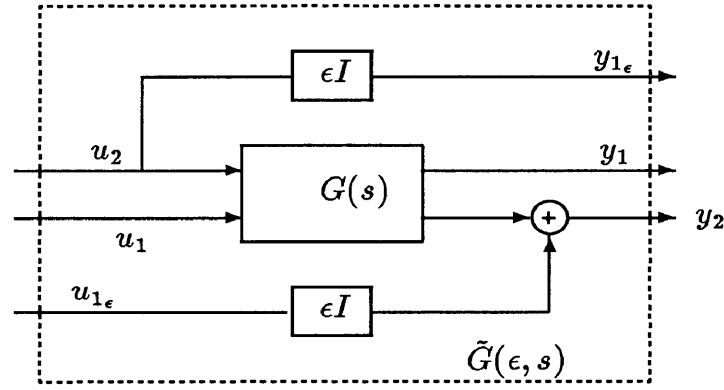


Figure 5.3: Modified Plant  $\tilde{G}(\epsilon, s)$

It is readily seen that the 2-Riccati procedure may be directly applied to  $\tilde{G}(\epsilon, s)$  since the corresponding problem is now regular. The relevant Riccati equations are

$$(A - B_2(D_{12}^T D_{12} + \epsilon^2 I)^{-1} D_{12}^T C_1)^T P_\epsilon + P_\epsilon (A - B_2(D_{12}^T D_{12} + \epsilon^2 I)^{-1} D_{12}^T C_1) + P_\epsilon (B_1 B_1^T - B_2(D_{12}^T D_{12} + \epsilon^2 I)^{-1} B_2^T) P_\epsilon + C_1^T (I - D_{12}(D_{12}^T D_{12} + \epsilon^2 I)^{-1} D_{12}^T) C_1 = 0 \quad (5.37)$$

$$(A - B_1 D_{21}^T (D_{21} D_{21}^T + \epsilon^2 I)^{-1} C_2) Q_\epsilon + Q_\epsilon (A - B_1 D_{21}^T (D_{21} D_{21}^T + \epsilon^2 I)^{-1} C_2)^T + Q_\epsilon (C_1^T C_1 - C_2^T (D_{21}^T D_{21} + \epsilon^2 I)^{-1} C_2) Q_\epsilon + B_1 (I - D_{21}^T (D_{21} D_{21}^T + \epsilon^2 I)^{-1} D_{21}) B_1^T = 0 \quad (5.38)$$

The following lemma gives necessary and sufficient conditions for the existence of a solution to the suboptimal problem for  $G(s)$ . We denote the closed loop transfer function from any input  $u$  to any output  $y$  for the augmented plant  $\tilde{G}(\epsilon, s)$  by  $T_{yu}(\epsilon, s)$ . The  $\epsilon$  argument will be dropped when the implied transfer function is independent of  $\epsilon$ .

**Lemma 5.4** *There exists a controller  $K(s)$  which solves the suboptimal  $H^\infty$  problem for the plant  $G(s)$  iff there exists an  $\epsilon^* > 0$  for which a solution to the suboptimal*

problem for the plant  $\tilde{G}(\epsilon^*, s)$  exists. Furthermore, if such an  $\epsilon^*$  exists, it holds that a solution to the problem exists  $\forall \epsilon \in [0, \epsilon^*]$ .

*Proof:* Suppose that there exists a compensator  $K(\epsilon^*, s)$  which solves the sub-optimal problem for  $\tilde{G}(\epsilon^*, s)$  for some  $\epsilon^* > 0$ . To prove sufficiency, it suffices to let the controller for each  $\epsilon \in [0, \epsilon^*]$  to be given by  $K(\epsilon, s) = K(s) := K(\epsilon^*, s)$ . Then

$\forall \epsilon$  we have

$$T_{\tilde{y}_1 \tilde{u}_1}(\epsilon, s) = \left[ \begin{array}{c|c} T_{y_1 u_1}(s) & \epsilon G_{12}(s) T_{22}(s) \\ \hline \epsilon T_{22} G_{21}(s) & \epsilon^2 T_{22}(s) \end{array} \right] \quad (5.39)$$

where

$$T_{y_1 u_1}(s) = \mathcal{F}_l(G(s), K(s)) \quad (5.40)$$

and

$$T_{22}(s) := K(s)(I - G_{22}(s)K(s))^{-1} \quad (5.41)$$

Since each partition in (5.39) is a stable transfer function at  $\epsilon = \epsilon^*$  it follows that  $T_{\tilde{y}_1 \tilde{u}_1}(\epsilon, s) \in RH^\infty \forall \epsilon$  and, in particular,  $\forall \epsilon \in [0, \epsilon^*]$ . By simple matrix manipulations, it follows that  $\forall \epsilon \in [0, \epsilon^*]$   $\|T_{\tilde{y}_1 \tilde{u}_1}(\epsilon, s)\|_\infty \leq \|T_{\tilde{y}_1 \tilde{u}_1}(\epsilon^*, s)\|_\infty < 1$ . In particular, we have that  $T_{y_1 u_1}(s) \in RH^\infty$ ,  $\|T_{y_1 u_1}(s)\|_\infty \leq \|T_{\tilde{y}_1 \tilde{u}_1}(\epsilon^*, s)\|_\infty < 1$ .

For necessity, assume that the controller  $K_0(s)$  solves the problem for the plant  $G(s)$ . Then we have that  $\|T_{y_1 u_1}\|_\infty = 1 - \delta$ ,  $1 > \delta > 0$ . Apply the controller  $K_0(s)$  and consider the closed loop system for increasing  $\epsilon$ . Since  $K_0(s)$  internally stabilizes  $\tilde{G}(0, s) \Rightarrow T_{y_1 u_1}, T_{22}, T_{22}G_{21}(s)$  and  $G_{12}T_{22}$  are all in  $RH^\infty$ ; moreover these matrices

do not depend on  $\epsilon \Rightarrow T_{\tilde{y}_1 \tilde{u}_1} \in RH^\infty \forall \epsilon$ . Hence we have that  $k_{12} := \|G_{12}T_{22}\|_\infty < \infty$ ,  
 $k_{21} := \|T_{22}G_{21}\|_\infty < \infty$ ,  $k_{22} := \|T_{22}\|_\infty < \infty$ . Moreover,

$$\|T_{\tilde{y}_1 \tilde{u}_1}(\epsilon, s)\|_\infty = \left\| \left[ \begin{array}{c|c} T_{y_1 u_1}(s) & \epsilon G_{12}(s)T_{22}(s) \\ \hline \epsilon T_{22}(s)G_{21}(s) & \epsilon^2 T_{22}(s) \end{array} \right] \right\|_\infty \quad (5.42)$$

$$\leq \|T_{y_1 u_1}\|_\infty + \epsilon k_{12} + \epsilon k_{21} + \epsilon^2 k_{22} \quad (5.43)$$

Now note that for  $\epsilon^* := \frac{\delta}{6 \max(k_{12}, k_{21}, \sqrt{k_{22}})}$ ,  $\|T_{\tilde{y}_1 \tilde{u}_1}(\epsilon, s)\|_\infty \leq 1 - \delta + \frac{\delta}{2} = 1 - \frac{\delta}{2} < 1$   
 $\forall \epsilon \in [0, \epsilon^*]$ .

□

Finally, we present the following lemma which will prove to be useful later on.

**Lemma 5.5** *Suppose that there exists an  $\epsilon^* > 0$  for which there is a solution to the suboptimal problem. For each  $\epsilon \in (0, \epsilon^*]$ , denote by  $P_\epsilon, Q_\epsilon$  the respective solutions of the Riccati equations (5.37,5.38). Then it holds that*

(i)  $P_0 := \lim_{\epsilon \rightarrow 0} P_\epsilon$  and  $Q_0 := \lim_{\epsilon \rightarrow 0} Q_\epsilon$  exist and  $P_0 \geq 0, Q_0 \geq 0$ .

(ii)  $\lambda_{\max}(Q_0 P_0) \leq 1$ .

To prove this lemma we again utilize the fact that the  $H^\infty$  problem can be tackled by solving the LQ differential game problem [36, 38] (5.19).

*Proof:* For (5.34) the cost in (5.19) is

$$J(\tilde{u}_1, u_2, \epsilon) = \int_0^\infty \tilde{y}_1^T \tilde{y}_1 - \tilde{u}_1^T \tilde{u}_1 dt \quad (5.44)$$

$$= \int_0^\infty y_1^T y_1 - u_1^T u_1 dt + \epsilon^2 \int_0^\infty u_2^T u_2 dt - \int_0^\infty u_{1\epsilon}^T u_{1\epsilon} dt \quad (5.45)$$

The optimizing inputs are [36, 38]

$$u_1^*(t) = B_1^T P_\epsilon x(t) \quad (5.46)$$

$$u_{1_\epsilon}^*(t) = 0 \quad (5.47)$$

$$u_2^*(t) = -(D_{12}^T D_{12} + \epsilon^2 I)^{-1} (D_{12}^T C_1 + B_2^T P_\epsilon) x(t) \quad (5.48)$$

Let  $y_1^*(t)$  be the output corresponding to  $u_1(t) = u_1^*(t)$ ,  $u_{1_\epsilon}(t) = 0$ ,  $u_2(t) = u_2^*(t)$  and initial condition  $x(0)$ . Now let  $\epsilon_1 > \epsilon_2 > 0$  and fix  $\epsilon = \epsilon_1$  in (5.45). Recall also that for any  $x(0) = x_0 \in \mathfrak{R}^n$  the optimal cost in (5.19) is

$$J^*(u_1^*, u_2^*, \epsilon) = x_0^T P_\epsilon x_0. \quad (5.49)$$

where  $P$  is a positive semidefinite matrix. It follows from the above that

$$x_0^T P_{\epsilon_2} x_0 \leq \int_0^\infty y_1^{*T} y_1^* - u_1^{*T} u_1^* + \epsilon_2^2 u_2^{*T} u_2^* dt \quad (5.50)$$

$$\leq x_0^T P_{\epsilon_1} x_0 \quad (5.51)$$

$\forall x_0 \in \mathfrak{R}^n$ . Hence, for any monotone decreasing sequence  $\{\epsilon_j\}_{j=0}^\infty$ ,  $\epsilon_j > 0$ ,  $P_{\epsilon_j}$  is a monotone decreasing sequence of non-negative definite (and therefore bounded) matrices; this implies that  $P_0 := \lim_{\epsilon \rightarrow 0} P_\epsilon$  exists and is also non-negative definite. By a similar argument,  $Q_0 := \lim_{\epsilon \rightarrow 0} Q_\epsilon$  exists and is non-negative definite. This proves Part (i).

Part (ii) follows from the fact that if the hypothesis of the Lemma holds, then

$$\lambda_{\max}(Q_\epsilon P_\epsilon) < 1 \quad \forall \epsilon \in (0, \epsilon^*] \Rightarrow \lambda_{\max}(Q_0 P_0) \leq 1. \quad \square$$

## 5.4 Solution to the Cheap $H^\infty$ Problem

In this section we consider the behavior of the regular problem (5.34) as  $\epsilon \rightarrow 0$ . Although the results of Section 5.3 rely on the special form (5.34), for ease of exposition, we shall simply represent the  $D_{12}$  and  $D_{21}$  matrices in (5.34) by  $D_{12}(\epsilon)$  and  $D_{21}(\epsilon)$  respectively. Also, let  $\mathcal{S}_{12}(\epsilon)$  be the stable eigenspace of the matrix pencil

$$W_{12}(\epsilon, s) := \begin{bmatrix} -sI + A & \tilde{B}_1 \tilde{B}_1^T & B_2 \\ -\tilde{C}_1^T \tilde{C}_1 & -sI - A^T & -\tilde{C}_1^T \tilde{D}_{12} \\ \tilde{D}_{12}^T \tilde{C}_1 & B_2^T & \tilde{D}_{12}^T \tilde{D}_{12} \end{bmatrix} \quad (5.52)$$

$$= \begin{bmatrix} -sI + A & B_1 B_1^T & B_2 \\ -C_1^T C_1 & -sI - A^T & -C_1^T D_{12} \\ D_{12}^T C_1 & B_2^T & D_{12}^T D_{12} + \epsilon^2 I_{r_2} \end{bmatrix} \quad (5.53)$$

$$(5.54)$$

Define also

$$M_{G_{12}}(s) := \left[ \begin{array}{c|c} -sI + A & B_2 \\ \hline C_1 & D_{12} \end{array} \right] \quad (5.55)$$

We now have the following theorem which characterizes the solution to the cheap  $H^\infty$  problem:

**Theorem 5.1** *Assume that  $C_2 = I_n$  and  $\tilde{D}_{12} = 0$  in (5.34) and that there is an*

*$\epsilon^* > 0$  for which there exists a solution to the  $H^\infty$  problem for  $\tilde{G}(\epsilon, s)$ . Let the*

*columns of  $\begin{bmatrix} X_{12,} \\ \Phi_{12,} \\ U_{12,} \end{bmatrix}$  form a basis for the stable eigenspace of  $W_{12}(s)$ . Let  $\begin{bmatrix} X_{12_{0\epsilon}}^{(l)} \\ U_{12_{0\epsilon}}^{(l)} \end{bmatrix}$*

be a matrix whose columns forms a basis for the sum of the finite  $C^{0\epsilon}$  zero eigenspace and the the lower infinite frequency eigensubspace of  $M_{G_{12}}(s)$ . Then

$$\lim_{\epsilon \rightarrow 0} S_{12}(\epsilon) = S_{12}(0) \cup \text{Im} \begin{bmatrix} X_{12_{0\epsilon}}^{(i)} \\ 0 \\ U_{12_{0\epsilon}}^{(i)} \end{bmatrix} \quad (5.56)$$

*Proof:* Note first of all that since a solution to the  $H^\infty$  problem for  $\tilde{G}(\epsilon, s)$  exists for some  $\epsilon = \epsilon^*$ , then by Lemmas 5.5 and 5.1  $W_{12}(\epsilon, s)$  has an  $n$ -dimensional stable eigenspace on  $[\epsilon^*, 0)$ . The proof then follows the same argument presented in the proof of Theorem 3.1. The following deviations should be noted:

(1) As in the previous section the cost index used here is that which pertains to the LQ differential game (5.19).

(2) As in [36] the “worst case”  $u_1$  is given by  $u_1^* = B_1^T \phi$  where  $\phi$  is obtained from the solution to  $W_{12}(0, s) \begin{pmatrix} x(s) \\ \phi(s) \\ u(s) \end{pmatrix} = \begin{pmatrix} x(0) \\ \phi(0) \\ 0 \end{pmatrix}$ . Note that by Lemma 3.2  $u_1^*$  is identically zero on the limiting infinite eigensubspace. Moreover, using a similar argument to that given in the proof to Theorem 3.1, it can be shown that, for any other subspace spanned by infinite eigenvectors of higher grade, there exist initial conditions for which  $J = \infty$ . The pertinent trajectories lie in the subspace spanned by the infinite eigenvectors of  $W_{12}(0, s)$  related to the highest grade infinite eigenvectors of  $M_{G_{12}}(s)$ . □



**Lemma 5.6** Let  $\mathcal{S}_{12}(0) = \text{Im} \begin{bmatrix} X_{12_s} \\ \Phi_{12_s} \\ U_{12_s} \end{bmatrix}$  and let  $\begin{bmatrix} X_{12_{oe}}^{(u)} \\ U_{12_{oe}}^{(u)} \end{bmatrix}$  be a matrix whose columns form a basis for the sum of the finite  $C^{0e}$  eigenspace and the upper infinite frequency eigensubspace of  $M_{G_{12}}(s)$ . Define  $\mathcal{X} = \begin{bmatrix} X_{12_s} & X_{12_{oe}}^{(u)} \end{bmatrix}$ . Then

(i)  $\text{Im } \mathcal{X}$  is the space of initial conditions  $x_0$  for which the cost (5.19) is finite.

(ii)  $\text{rank } \mathcal{X} < n \Rightarrow$  no solution to the  $H^\infty$  problem for  $G(s)$  exists.

*Proof:* Follows the same arguments as for the proof to Lemma 3.8. Note that in this case the nonsingularity of  $\mathcal{X}$  is not enough to guarantee that a solution to the problem exists. □

**Theorem 5.2** Let  $\begin{bmatrix} X_{12_s} \\ \Phi_{12_s} \\ U_{12_s} \end{bmatrix}$ ,  $\begin{bmatrix} X_{12_{oe}}^{(u)} \\ U_{12_{oe}}^{(u)} \end{bmatrix}$  and  $\mathcal{X}$  be as defined in Lemma 5.6. Assume that  $\text{rank } \mathcal{X} = n$ , and that there exists an  $\epsilon^* > 0$  for which there is a solution to the  $H^\infty$  problem for  $\tilde{G}(\epsilon^*, s)$ . Then the limiting solution  $P_0$  to the Riccati equation (5.37) is the positive semidefinite matrix

$$P_0 = \Phi_{12_s} H_{12_s}^T \tag{5.57}$$

where

$$\begin{bmatrix} H_{12_s}^T \\ H_{12_{oe}}^T \end{bmatrix} \begin{bmatrix} X_{12_s} & X_{12_{oe}} \end{bmatrix} = I \tag{5.58}$$

*Proof:* The proof is a slight modification of the proof to Theorem 3.2. First note that by Lemma 5.6, the existence of a solution to the  $H^\infty$  problem implies that  $\forall x_0 \in \mathfrak{R}^n \exists$  vectors  $v, v_{oe}$  such that

$$x_0 = X_{12} v + X_{oe}^{(u)} v_{oe} \quad (5.59)$$

Moreover, the resulting cost is equivalent to that obtained for  $x_0 = X_{12} v$  since the cost pertaining to the  $C^{oe}$  eigenspace of  $W_{12}(0, s)$  is zero. In the time domain the optimal state and control trajectories corresponding to  $x_0$  is given by  $\begin{pmatrix} x(t) \\ u_2(t) \end{pmatrix} =$

$\begin{bmatrix} X_{12} \\ U_{12} \end{bmatrix} e^{\Lambda_{12} t} v$  for  $t > 0$ , where  $\Lambda_{12}$  is strictly Hurwitz and satisfies

$$\begin{bmatrix} A & B_1 B_1^T & B_2 \\ -C_1^T C_1 & -A^T & -C_1^T D_{12} \\ D_{12}^T C_1 & B_2^T & D_{12}^T D_{12} \end{bmatrix} \begin{bmatrix} X_{12} \\ \Phi_{12} \\ U_{12} \end{bmatrix} = \begin{bmatrix} X_{12} \\ \Phi_{12} \\ 0 \end{bmatrix} \Lambda_{12} \quad (5.60)$$

The optimal cost is thus given by (Lemma 5.2)

$$\begin{aligned} J(x_0) &= \int_0^\infty \begin{pmatrix} x \\ u_2 \end{pmatrix}^T \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix} \begin{pmatrix} x \\ u_2 \end{pmatrix} - u_1^T u_1 dt \\ &= v^T \left( \int_0^\infty e^{\Lambda^T t} \begin{bmatrix} X \\ U \end{bmatrix}^T \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} - \Phi_{12}^T B_1 B_1^T \Phi_{12} \right) e^{\Lambda t} dt v \quad (5.61) \end{aligned}$$

Now from (5.60) it can be shown that

$$\begin{bmatrix} X_{12} \\ U_{12} \end{bmatrix}^T \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix} \begin{bmatrix} X_{12} \\ U_{12} \end{bmatrix} \Phi_{12}^T B_1 B_1^T \Phi_{12} = - (X_{12}^T \Phi_{12} \Lambda_{12} + \Lambda_{12}^T X_{12}^T \Phi_{12}) \quad (5.62)$$

Hence we have that

$$\begin{aligned}
J(x_0) &= -v^T \left( \int_0^\infty e^{\Lambda_{12_s}^T t} (X_{12_s}^T \Phi_{12_s} \Lambda_{12_s} + \Lambda_{12_s}^T X_{12_s}^T \Phi_{12_s}) e^{\Lambda_{12_s} t} dt \right) v \\
&= -v^T \left( \int_0^\infty \frac{d}{dt} (e^{\Lambda_{12_s}^T t} X_{12_s}^T \Phi_{12_s} e^{\Lambda_{12_s} t}) dt \right) v \\
&= v^T X_{12_s}^T \Phi_{12_s} v
\end{aligned} \tag{5.63}$$

the last equality following from the fact that  $\Lambda_{12_s}$  is Hurwitz. Since  $\begin{bmatrix} X_{12_s} & X_{0e}^{(u)} \end{bmatrix}$  is nonsingular  $\Rightarrow \exists$  a matrix  $\begin{bmatrix} H_{12_s} & H_{oe} \end{bmatrix}$  such that

$$\begin{bmatrix} H_{12_s}^T \\ H_{oe}^T \end{bmatrix} \begin{bmatrix} X_{12_s} & X_{0e}^{(u)} \end{bmatrix} = I_n. \tag{5.64}$$

Left multiply (5.59) by  $H_{12_s}^T$  to obtain

$$v = H_{12_s}^T x_0 \tag{5.65}$$

and substitute in (5.63) to get

$$J(x_0) = x_0^T H_{12_s} X_{12_s}^T \Phi_{12_s} H_{12_s}^T x_0 \tag{5.66}$$

Now we note that from Lemma 3.3 that  $X_{0e}^T \Phi_{12_s} = 0 \Rightarrow H_{oe} X_{0e}^T \Phi_{12_s} = 0 \Rightarrow$

$$(H_{12_s} X_{12_s}^T - I) \Phi = 0 \tag{5.67}$$

the last relationship following from (5.64). Rewrite (5.67) as  $H X^T \Phi = \Phi$  and substitute in (5.66) to give

$$\begin{aligned}
J &= x_0^T \Phi_{12_s} H_{12_s}^T x_0 \\
&= x_0^T P_* x_0
\end{aligned} \tag{5.68}$$

Since  $J(u_1^*, u_2^*, x(0)) \geq 0$  for all  $\epsilon$  it follows that  $P_*$  is indeed a matrix of a positive semi-definite form. To show that it is also symmetric define  $N := \Phi_{12,}^T X_{12,} - X_{12,}^T \Phi_{12,}$  and pre-multiply both sides of (5.60) by  $\begin{bmatrix} \Phi_{12,}^T & -X_{12,}^T & 0 \end{bmatrix}$  to obtain

$$N\Lambda = \Phi_{12,}^T A X_{12,} + (\Phi_{12,}^T A X_{12,})^T + X_{12,}^T C^T C X_{12,} - U_{12,}^T D^T D U_{12,} + \Phi_{12,}^T B_1 B_1^T \Phi_{12,} \quad (5.69)$$

Since the RHS of (5.69) is symmetric and  $N$  is skew symmetric, it follows that

$$(-\Lambda_{12,}^T)N - N\Lambda_{12,} = 0 \quad (5.70)$$

Since  $\Lambda_{12,}$  is Hurwitz, and the eigenvalues of  $-\Lambda_{12,}^T$  and  $\Lambda_{12,}$  are disjoint  $\Rightarrow$  by Lemma 1.5 [31] that the unique solution to the above is  $N = 0 \Rightarrow \Phi_{12,}^T X_{12,} = X_{12,}^T \Phi_{12,}$ .  $\square$

Before proceeding to the main result of this section, let us define the system matrix

$$M_{G_{21}}(s) := \left[ \begin{array}{c|c} -sI + A^T & C_2^T \\ \hline B_1^T & D_{21}^T \end{array} \right] \quad (5.71)$$

Let  $\begin{bmatrix} X_{12\infty}^{(l)} \\ U_{12\infty}^{(l)} \end{bmatrix}$  and  $\begin{bmatrix} X_{21\infty}^{(l)} \\ U_{21\infty}^{(l)} \end{bmatrix}$  be matrices whose columns form bases for the lower infinite zero eigensubspaces of  $M_{G_{12}}(s)$  and  $M_{G_{21}}(s)$  respectively; similarly, let

$\begin{bmatrix} X_{12\infty}^{(u)} \\ U_{12\infty}^{(u)} \end{bmatrix}$

and  $\begin{bmatrix} X_{21\infty}^{(u)} \\ U_{21\infty}^{(u)} \end{bmatrix}$  be matrices whose columns form bases for the upper infinite zero eigensubspaces of the respective pencils  $M_{G_{12}}(s)$  and  $M_{G_{21}}(s)$ .

In addition, it will be assumed that the columns of  $\begin{bmatrix} X_{12,j} \\ U_{12,j} \end{bmatrix}$  and  $\begin{bmatrix} X_{21,j} \\ U_{21,j} \end{bmatrix}$  form bases for the finite  $C^{0\epsilon}$  eigenspaces of  $M_{G_{12}}(s)$  and  $M_{G_{21}}(s)$  respectively, corresponding to the eigenvalues of the matrices  $\Lambda_{12,j}$  and  $\Lambda_{21,j}$ , i. e.

$$M_{G_{12}}(s) \begin{bmatrix} X_{12,j} \\ U_{12,j} \end{bmatrix} = \begin{bmatrix} X_{12,j} \\ 0 \end{bmatrix} (-sI + \Lambda_{12,j}) \quad (5.72)$$

and

$$M_{G_{21}}(s) \begin{bmatrix} X_{21,j} \\ U_{21,j} \end{bmatrix} = \begin{bmatrix} X_{21,j} \\ 0 \end{bmatrix} (-sI + \Lambda_{21,j}) \quad (5.73)$$

To set up the appropriate eigenproblems, we define  $W_{12}(\epsilon, s)$  as before and

$$W_{21}(\epsilon, s) := \begin{bmatrix} -sI + A^T & B_1 B_1^T & C_2^T \\ -B_1 B_1^T & -sI - A & -B_1 D_{21}^T \\ D_{21} B_1^T & C_2 & D_{21} D_{21}^T + \epsilon^2 I_{m_2} \end{bmatrix} \quad (5.74)$$

Now let the columns of  $\begin{bmatrix} X_{12,s} \\ \Phi_{12,s} \\ U_{12,s} \end{bmatrix}$  and  $\begin{bmatrix} X_{21,s} \\ \Phi_{21,s} \\ U_{21,s} \end{bmatrix}$  form bases for the stable subspaces of  $W_{12}(0, s)$  and  $W_{21}(0, s)$  respectively, corresponding to the eigenvalues of the matrices  $\Lambda_{12,s}$  and  $\Lambda_{21,s}$ , i. e.

$$W_{12}(0, s) \begin{bmatrix} X_{12,s} \\ \Phi_{12,s} \\ U_{12,s} \end{bmatrix} = \begin{bmatrix} X_{12,s} \\ \Phi_{12,s} \\ 0 \end{bmatrix} (-sI + \Lambda_{12,s}) \quad (5.75)$$

and

$$W_{21}(0, s) \begin{bmatrix} X_{21,s} \\ \Phi_{21,s} \\ U_{21,s} \end{bmatrix} = \begin{bmatrix} X_{21,s} \\ \Phi_{21,s} \\ 0 \end{bmatrix} (-sI + \Lambda_{21,s}) \quad (5.76)$$

Finally define

$$\Xi_{12} := \begin{bmatrix} X_{12} \\ \Phi_{12} \\ U_{12} \end{bmatrix} = \begin{bmatrix} X_{12,} & X_{12,j} & X_{12,\infty}^{(l)} \\ \Phi_{12,} & 0 & 0 \\ U_{12,} & U_{12,j} & U_{12,\infty}^{(l)} \end{bmatrix} \quad (5.77)$$

$$\Xi_{21} := \begin{bmatrix} X_{21} \\ \Phi_{21} \\ U_{21} \end{bmatrix} = \begin{bmatrix} X_{21,} & X_{21,j} & X_{21,\infty}^{(l)} \\ \Phi_{21,} & 0 & 0 \\ U_{21,} & U_{21,j} & U_{21,\infty}^{(l)} \end{bmatrix} \quad (5.78)$$

and note that for all  $\epsilon > 0$  the central controller  $K_0(\epsilon, s)$  has system matrix representation

$$M_{K_0}(\epsilon, s) = \left[ \begin{array}{c|c} \Xi_{21}^T(\epsilon) \begin{bmatrix} -sI + A & B_1 B_1^T & B_2 \\ C_1^T C_1 & sI + A^T & C_1^T \tilde{D}_{12}(\epsilon) \\ C_2 & \tilde{D}_{21}(\epsilon) B_1^T & 0 \end{bmatrix} & \begin{bmatrix} \Xi_{12}(\epsilon) \\ U_{21}^T(\epsilon) \end{bmatrix} \\ \hline -U_{12}(\epsilon) & 0 \end{array} \right] \quad (5.79)$$

where the columns of  $\Xi_{12}(\epsilon) := \begin{bmatrix} X_{12}(\epsilon) \\ \Lambda_{12}(\epsilon) \\ U_{12}(\epsilon) \end{bmatrix}$  and  $\Xi_{21}(\epsilon) := \begin{bmatrix} X_{21}(\epsilon) \\ \Lambda_{21}(\epsilon) \\ U_{21}(\epsilon) \end{bmatrix}$  form bases for the respective stable eigenspaces of  $W_{21}(\epsilon, s)$  and  $W_{12}(\epsilon, s)$ , i. e.

$$W_{12}(\epsilon, s) \begin{bmatrix} X_{12}(\epsilon) \\ \Phi_{12}(\epsilon) \\ U_{12}(\epsilon) \end{bmatrix} = \begin{bmatrix} X_{12}(\epsilon) \\ \Phi_{12}(\epsilon) \\ 0 \end{bmatrix} (-sI + \Lambda_{12}(\epsilon)) \quad (5.80)$$

and

$$W_{21}(\epsilon, s) \begin{bmatrix} X_{21}(\epsilon) \\ \Phi_{21}(\epsilon) \\ U_{21}(\epsilon) \end{bmatrix} = \begin{bmatrix} X_{21}(\epsilon) \\ \Phi_{21}(\epsilon) \\ 0 \end{bmatrix} (-sI + \Lambda_{21}(\epsilon)) \quad (5.81)$$

where  $\Lambda_{12}(\epsilon)$  and  $\Lambda_{21}(\epsilon)$  are asymptotically stable matrices  $\forall \epsilon > 0$ .

The limiting form of the required compensator is given by the following

**Theorem 5.3** Suppose that  $\exists \epsilon = \epsilon^* > 0$  for which a solution to the  $H^\infty$  problem for (5.34) exists.

Then, for  $\epsilon \rightarrow 0$  the following hold:

(i) The limiting value of the controller system matrix is given by

$$M_{K_0}(s) := \left[ \begin{array}{ccc|cc} \Xi_{21}^T & \begin{bmatrix} -sI + A & B_1 B_1^T & B_2 \\ C_1^T C_1 & sI + A^T & C_1^T D_{12} \\ C_2 & D_{21} B_1^T & 0 \end{bmatrix} & \Xi_{12} & U_{21}^T \\ \hline & -U_{12} & & 0 \end{array} \right]. \quad (5.82)$$

(ii) The limiting closed loop transfer function,  $T_{y_1 u_1}(s)$ , is asymptotically stable with  $\|T_{y_1 u_1}\| \leq 1$  and is given by

$$T_{y_1 u_1}(s) = \mathcal{F}_l(G(s), K_0(s)) \stackrel{s}{=} \left[ \begin{array}{ccc|c} -sI + \Lambda_{12s} - H_{12s}^T B_1 B_1^T \bar{\kappa}_{12s} & H_{12s}^T (-sI + A) H_{21s} & H_{12s}^T B_1 & \\ \bar{\kappa}_{21s}^T (-sI + A) \bar{\kappa}_{12s} + \bar{\kappa}_{21s}^T \bar{\kappa}_{12s} \Lambda_{12s} + \Lambda_{21s}^T \bar{\kappa}_{21s}^T \bar{\kappa}_{12s} & -sI + \Lambda_{21s}^T - \bar{\kappa}_{21s}^T C_1^T C_1 H_{21s} & X_{21s}^T B_1 + U_{21s}^T D_{21} & \\ \hline C_1 X_{12s} + D_{12} U_{12s} & C_1 H_{21s} & 0 & \end{array} \right] \quad (5.83)$$

Where  $H_{12s}$ ,  $H_{21s}$  satisfy

$$H_{12}^T := \begin{bmatrix} H_{12s}^T \\ H_{12j}^T \\ H_{12\infty}^T \end{bmatrix} = \begin{bmatrix} X_{12s} & X_{12j} & X_{12\infty}^{(u)} \end{bmatrix}^{-1} \quad (5.84)$$

$$H_{21}^T := \begin{bmatrix} H_{21_i}^T \\ H_{21_j}^T \\ H_{21_\infty}^T \end{bmatrix} = \begin{bmatrix} X_{21_i} & X_{21_j} & X_{21_\infty}^{(u)} \end{bmatrix}^{-1}. \quad (5.85)$$

(iii) If the system matrix  $M_{K_0}(s)$  is well-posed, then  $\lim_{\epsilon \rightarrow 0} K_0(\epsilon, s)$  exists and is given by

$$K_0(s) \stackrel{\text{a}}{=} M_{K_0}(s) \quad (5.86)$$

and this value of  $K_0(s)$  achieves the limiting transfer function  $T_{y_1 u_1}(s)$ .

*Proof:* As noted before, for  $\epsilon > 0$  the central controller has system matrix given by (5.79). The fact that  $M_{K_0}(\epsilon, s) \rightarrow M_{K_0}(s)$  follows from Theorem 5.1 and the analyticity of  $\Xi_{12}(\epsilon)$ ,  $\Xi_{21}(\epsilon)$ .

For part (ii) assume that  $\epsilon^*$  is small enough so that those stable eigenvalues of  $W_{12}(\epsilon, s)$  which approach locations at infinity, the finite  $j\omega$ -axis and the open LHP are all in disjoint groups on  $[0, \epsilon^*]$ . For  $\epsilon > 0$  we know that the closed loop transfer function from  $u_1$  to  $y_1$  has Rosenbrock system matrix

$$M_{T_{y_1 u_1}}(\epsilon, s) = \left[ \begin{array}{cc|c} -sI + A & -B_2 U_{12}(\epsilon) & B_1 \\ U_{21}^T(\epsilon) & -sE_k + A_k & -U_{21}^T(\epsilon) D_{21} \\ \hline C_1 & -D_{12} U_{12}(\epsilon) & 0 \end{array} \right] \quad (5.87)$$

where  $E_k, A_k$  are given in (5.28). For  $\epsilon < \epsilon^*$  we can partition  $\Xi_{12}(\epsilon)$  and  $\Xi_{21}(\epsilon)$  as follows:



$$\Xi_{12}(\epsilon) = \left[ \begin{array}{c|c|c} X_{12_s}(\epsilon) & X_{12_j}(\epsilon) & X_{12_\infty}(\epsilon) \\ \Phi_{12_s}(\epsilon) & \Phi_{12_j}(\epsilon) & \Phi_{12_\infty}(\epsilon) \\ U_{12_s}(\epsilon) & U_{12_j}(\epsilon) & U_{12_\infty}(\epsilon) \end{array} \right] \quad (5.88)$$

$$\Xi_{21}(\epsilon) = \left[ \begin{array}{c|c|c} X_{21_s}(\epsilon) & X_{21_j}(\epsilon) & X_{21_\infty}(\epsilon) \\ \Phi_{21_s}(\epsilon) & \Phi_{21_j}(\epsilon) & \Phi_{21_\infty}(\epsilon) \\ U_{21_s}(\epsilon) & U_{21_j}(\epsilon) & U_{21_\infty}(\epsilon) \end{array} \right] \quad (5.89)$$

where

$$\lim_{\epsilon \rightarrow 0} \left[ \begin{array}{c|c|c} X_{12_s}(\epsilon) & X_{12_j}(\epsilon) & X_{12_\infty}(\epsilon) \\ \Phi_{12_s}(\epsilon) & \Phi_{12_j}(\epsilon) & \Phi_{12_\infty}(\epsilon) \\ U_{12_s}(\epsilon) & U_{12_j}(\epsilon) & U_{12_\infty}(\epsilon) \end{array} \right] = \left[ \begin{array}{c|c|c} X_{12_s} & X_{12_j} & X_{12_\infty}^{(I)} \\ \Phi_{12_s} & \Phi_{12_j} & 0 \\ U_{12_s} & U_{12_j} & U_{12_\infty}^{(I)} \end{array} \right] \quad (5.90)$$

and

$$\lim_{\epsilon \rightarrow 0} \left[ \begin{array}{c|c|c} X_{21_s}(\epsilon) & X_{21_j}(\epsilon) & X_{21_\infty}(\epsilon) \\ \Phi_{21_s}(\epsilon) & \Phi_{21_j}(\epsilon) & \Phi_{21_\infty}(\epsilon) \\ U_{21_s}(\epsilon) & U_{21_j}(\epsilon) & U_{21_\infty}(\epsilon) \end{array} \right] = \left[ \begin{array}{c|c|c} X_{21_s} & X_{21_j} & X_{21_\infty}^{(I)} \\ \Phi_{21_s} & \Phi_{21_j} & 0 \\ U_{21_s} & U_{21_j} & U_{21_\infty}^{(I)} \end{array} \right] \quad (5.91)$$

Associated with those zeros of interest which remain finite as  $\epsilon \rightarrow 0$  are the matrices  $\Lambda_{12_s}(\epsilon)$ ,  $\Lambda_{12_j}(\epsilon)$  whose eigenvalues correspond to the eigenvalues of  $W(\epsilon, s)$  on each subspace pertaining to the closed right half plane; similarly,  $\Lambda_{12_s}(\epsilon)$ ,  $\Lambda_{12_j}(\epsilon)$  are matrices whose eigenvalues correspond to those of  $W_{21}(\epsilon, s)$  on the pertinent subspaces. Moreover, we have that

$$\lim_{\epsilon \rightarrow 0} \Lambda_{12_s}(\epsilon) = \Lambda_{12_s}, \quad \lim_{\epsilon \rightarrow 0} \Lambda_{12_j}(\epsilon) = \Lambda_{12_j} \quad (5.92)$$

$$\lim_{\epsilon \rightarrow 0} \Lambda_{21_s}(\epsilon) = \Lambda_{21_s}, \quad \lim_{\epsilon \rightarrow 0} \Lambda_{21_j}(\epsilon) = \Lambda_{21_j} \quad (5.93)$$

Substituting and partitioning in (5.87) we obtain

$$\begin{array}{l}
M_{T_{y_1 u_1}}(\epsilon, s) = \\
\begin{array}{|l}
-sI + A \\
U_{21}^T C_2 \\
U_{21\infty}^T C_2 \\
C_1
\end{array}
\begin{array}{|l}
-B_2 U_{12} \\
-B_2 U_{12} \\
-B_2 U_{12} \\
-D_{12} U_{12} \\
-D_{12} U_{12} \\
-D_{12} U_{12} \\
-D_{12} U_{12}
\end{array}
\begin{array}{|l}
-B_2 U_{12\infty} \\
X_{21}^T [(-sI + A)X_{12} + B_2 U_{12}] \\
\Phi_{21}^T C_1^T (C_1 X_{12} + D_{12} U_{12}) \\
+ X_{21}^T B_1 B_1^T \Phi_{12} + \Phi_{21}^T C_1^T C_1 X_{12} \\
+ X_{21}^T B_2 U_{12} + U_{21}^T C_2 X_{12} \\
+ \Phi_{21}^T C_1^T D_{12} U_{12} + U_{21}^T D_{21} B_1^T \Phi_{12} \\
+ [X_{21}^T B_1 + U_{21}^T D_{21}] B_1^T \Phi_{12} \\
+ U_{21}^T C_2 X_{12} \\
X_{21}^T [(-sI + A)X_{12} + B_2 U_{12}] \\
+ [X_{21}^T B_1 + U_{21}^T D_{21}] B_1^T \Phi_{12} \\
+ U_{21}^T C_2 X_{12} \\
X_{21\infty}^T [(-sI + A)X_{12} + B_2 U_{12}] \\
+ [X_{21\infty}^T B_1 + U_{21\infty}^T D_{21}] B_1^T \Phi_{12} \\
+ U_{21\infty}^T C_2 X_{12} \\
-D_{12} U_{12} \\
-D_{12} U_{12} \\
-D_{12} U_{12} \\
-D_{12} U_{12}
\end{array}
\begin{array}{|l}
B_1 \\
-U_{21}^T D_{21} \\
-U_{21\infty}^T D_{21} \\
0
\end{array}
\end{array}
\tag{5.94}$$

Now take the limit as  $\epsilon \rightarrow 0$  to get

$$M_{T_{y_1 u_1}}(s) = \lim_{\epsilon \rightarrow 0} M_{T_{y_1 u_1}}(\epsilon, s) \quad (5.95)$$

Perform the following transformations

$$\text{column 2} = \text{column 2} - \text{column 1} \times X_{12_s}$$

$$\text{column 3} = \text{column 3} - \text{column 1} \times X_{12_j}$$

$$\text{column 4} = \text{column 4} - \text{column 1} \times X_{12_\infty}$$

$$\text{row 2} = \text{row 2} + X_{21_s}^T \times \text{row 1}$$

$$\text{row 3} = \text{row 3} + X_{21_j}^T \times \text{row 1}$$

$$\text{row 4} = \text{row 4} + X_{21_\infty}^T \times \text{row 1}$$

$$\text{row 1} = H_{12}^T \times \text{row 1}$$

$$\text{column 1} = \text{column 1} \times H_{21}$$

where  $H_{12}$  and  $H_{21}$  are as given in the theorem. Partition the first column and row in conformity with the partitioning of  $H_{21}$  and  $H_{12}^T$  and rearrange columns and rows to get

$$M_{T_{y_1} u_1}(s) = \begin{bmatrix} H_{12}^T(-sI + A)H_{21} & -(-sI + \Lambda_{12}) + H_{12}^T B_1 B_1^T \Phi_{12} & 0 & 0 & * & H_{12}^T B_1 \\ (-sI + \Lambda_{21}^T) - \Phi_{21}^T C_1^T C_1 H_{21} & \Theta & 0 & 0 & * & X_{21}^T B_1 + U_{21}^T D_{21} \\ * & * & H_{12}^T X_{12} s - I & 0 & * & H_{12}^T B_1 \\ * & * & 0 & -sI + \Lambda_{12} & * & H_{12}^T B_1 \\ 0 & 0 & 0 & 0 & -sI + \Lambda_{21}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & X_{21}^T H_{21} s - I \\ C_1 H_{21} & -(C_1 X_{12} + D_{12} U_{12}) & 0 & 0 & C_1 H_{21} & C_1 H_{21} \\ & & & & & 0 \end{bmatrix} \quad (5.96)$$

where

$$\Theta = \Phi_{21}^T(-sI + A)\Phi_{12} + \Phi_{21}^T\Phi_{12}\Lambda_{12} + \Lambda_{21}^T\Phi_{21}^T\Phi_{12}, \quad (5.97)$$

We can now eliminate all but the first two rows and columns in the (1,1) block since the remaining pertain to diagonal blocks which are either completely unobservable or completely uncontrollable. This gives the result in (5.83). Note that the transfer functions  $T_{y_1 u_1}(\epsilon, s)$  are all  $RH^\infty$  functions with less than unit norm. It therefore follows that  $T_{y_1 u_1} \in RH^\infty$  with  $\|T_{y_1 u_1}\| \leq 1$

For part (iii) simply note that  $\mathcal{F}_l(G(s), K_0(s)) \stackrel{s}{=} M_{T_{y_1 u_1}}(s)$  in (5.95).

□

*Comment:* As in the LQG case, the limiting controller may not exist. This is shown by example in the following section. This situation may be shown to arise from the fact that there is an “almost disturbance decoupling problem” embedded in the singular  $H^\infty$  problem [57, 56].

*Comment:* For singular problems the matrix  $E_k$  will be rank deficient. This follows from the fact that the rank deficiency of  $\begin{bmatrix} X_{12} \\ \Phi_{12} \end{bmatrix}$  is  $r_2$  since the  $r_2$  grade 1 infinite eigenvectors of  $M_{G_{12}}(s)$  are of the form  $\begin{bmatrix} 0 \\ u \end{bmatrix}$ . A similar result holds for  $M_{G_{21}}(s)$ . This indicates that, in general, the central controller  $K_0(s)$  may be improper.

*Comment:* By Theorem 5.1, the limiting solution to (5.38) is given by

$$Q_0 = H_{21} \Phi_{21}^T. \quad (5.98)$$

Theorem 5.3 gives the limiting form for the  $H^\infty$  controller, if indeed the limit does exist. If this is the case, then it may be improper; this is seen in the examples described in the following section. There are cases, however, when the limiting compensator is proper and is also of reduced order, i. e. of order  $< n$ . This has been noted in [27] and is the subject of the following corollary to Theorem 5.3:

**Corollary 5.1 (Reduced Order Proper Controller)** *Let  $r_{2_1} := \text{rank } D_{12}$  and  $r_{2_2} := r_2 - r_{2_1}$ . Assume, without loss of generality that*

$$D_{12} = \begin{bmatrix} D_{12_1} & 0 \end{bmatrix} \quad (5.99)$$

where  $D_{12_1} \in \mathbb{R}^{r_2 \times r_{2_1}}$  is injective and partition  $B_2$  in conformity with the right hand side of (5.99), i. e.

$$B_2 = \begin{bmatrix} B_{2_1} & B_{2_2} \end{bmatrix} \quad (5.100)$$

Let  $D_{11} = 0$ ,  $D_{22} = 0$ ,  $D_{21}$  be surjective and  $C_1 B_{2_2}$  be injective. Suppose that  $W_{12}(s)$  and  $W_{21}(s)$  have no finite  $C^{0e}$  zeros and that there exists an  $\epsilon^* > 0$  for which there is a solution to the  $H^\infty$  problem for  $\tilde{G}(\epsilon^*, s)$  with  $\lambda_{\max}(Q_0 P_0) < 1$ . Then there exists a proper controller  $K(s)$  which solves the cheap  $H^\infty$  problem for  $G(s)$ ; the order of this controller is at most  $n - r_{2_2}$ .

*Proof:* The condition  $C_1 B_{2_2}$  injective implies that  $M_{G_{12}}$  has a simple infinite zero structure, i. e. all infinite zeros of  $M_{G_{12}}(s)$  are of order unity.

The infinite zero eigenvectors are then the columns of  $\begin{bmatrix} 0 & B_{2_2} \\ \hat{I}_{r_{2_2}} & 0 \end{bmatrix}$  and  $\Xi_{12} =$

$$\begin{bmatrix} X_{12_s} & 0 \\ \Phi_{12_s} & 0 \\ U_{12_s} & \hat{I}_{r_{2_2}} \end{bmatrix}$$

where,

$$\hat{I}_{r_{2_2}} := \begin{bmatrix} 0_{r_{2_1} \times r_{2_2}} \\ I_{r_{2_2}} \end{bmatrix} \quad (5.101)$$

Suppose that  $\Xi_{21} = \begin{bmatrix} X_{21} \\ \Phi_{21} \\ U_{21} \end{bmatrix}$ ,  $\Xi_{21} \in \mathfrak{R}^{(2n+m) \times n}$ . Then if a solution exists  $\Rightarrow$  by

Theorem 5.1 that  $\begin{bmatrix} X_{12_s} & B_{2_2} \end{bmatrix}$  and  $X_{21}$  are nonsingular and

$$P_0 = \begin{bmatrix} \Phi_{12_s} & 0 \end{bmatrix} \begin{bmatrix} X_{12_s} & B_{2_2} \end{bmatrix}^{-1} \geq 0 \quad (5.102)$$

$$Q_0 = X_{21}^{-1T} \Phi_{21}^T \geq 0 \quad (5.103)$$

We are given that  $\lambda_{\max} Q_0 P_0 < 1 \Rightarrow$

$$\lambda_{\max} \left( -s X_{21}^T \begin{bmatrix} X_{12_s} & B_{2_2} \end{bmatrix} + \begin{bmatrix} \Phi_{21}^T \Phi_{12_s} & 0 \end{bmatrix} \right) < 1 \quad (5.104)$$

$\Rightarrow$  by taking  $s = 1$

$$\text{rank} \left[ \begin{array}{c|c} X_{21}^T X_{12_s} - \Phi_{21}^T \Phi_{12_s} & X_{21}^T B_{2_2} \end{array} \right] = n \quad (5.105)$$

Substitution in (5.27) gives

$$K_0(s) \stackrel{\text{def}}{=} \left[ \begin{array}{c|c|c} (-sI + \Lambda_{21}^T) \begin{bmatrix} X_{21}^T X_{12_s} - \Phi_{21}^T \Phi_{12_s} & X_{21}^T B_{2_2} \end{bmatrix} + X_{21}^T B_{2_2} U_{12_s} & X_{21}^T B_{2_2} & U_{21}^T \\ \hline U_{12_s} & \hat{I}_{r_{2_2}} & 0 \end{array} \right] \quad (5.106)$$

Let  $T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$  be a row compression on  $X_{21}^T X_{12_s} - \Phi_{21}^T \Phi_{12_s}$ . Then from (5.105)

$$T \begin{bmatrix} -(X_{21}^T X_{12_s} - \Phi_{21}^T \Phi_{12_s}) & X_{21}^T B_{2_2} \end{bmatrix} = \begin{bmatrix} E_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad (5.107)$$

where  $E_{11} \in \mathfrak{R}^{(n-r) \times (n-r)}$  and  $A_{22} \in \mathfrak{R}^{r \times r}$  are nonsingular. By simple equivalence operations, it can be shown that

$$K_0(s) \stackrel{s}{=} \left[ \begin{array}{cc|c} -sE_{11} + A_{11} & A_{12} & T_1 U_{21} \\ A_{21} & A_{22} & T_2 U_{21} \\ \hline U_{12_s} & \hat{I}_{r_{2_2}} & 0 \end{array} \right] \quad (5.108)$$

$$\cong \left[ \begin{array}{cc|c} -sI + E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21}) & E_{11}^{-1}(T_1 - A_{12}A_{22}^{-1}T_2)U_{21}^T \\ \hline U_{12_s} - \hat{I}_{r_{2_2}}A_{22}^{-1}A_{21} & -\hat{I}_{r_{2_2}}A_{22}^{-1}T_2U_{21}^T \end{array} \right]. \quad (5.109)$$

where

$$A_{11} := T_1(\Lambda_{21}^T[X_{21}^T X_{12_s} - \Phi_{21}^T \Phi_{12_s}] + X_{21}^T B_2 U_{12_s}) \quad (5.110)$$

and

$$A_{21} := T_2(\Lambda_{21}^T[X_{21}^T X_{12_s} - \Phi_{21}^T \Phi_{12_s}] + X_{21}^T B_2 U_{12_s}) \quad (5.111)$$

which is a realization of  $K_0(s)$  of order at most  $n - r_{2_2}$ .  $\square$

*Comment:* It should be noted that Lemma 5.1 cannot be readily extended to the case where  $M_{G_{12}}(s)$  has a more complex zero structure at infinity. If for example all

of the infinite zeros were of order 2, then a similar analysis to that carried out above reveals that  $A_{22} = 0$  in (5.107). This indicates that if the pencil  $\begin{bmatrix} -sE_{11} + A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$  is regular then, in general,  $K_0(s)$  in (5.108) would be improper. It can also be shown



that for this controller, the modes at infinity are uncontrollable (see [3] for this notation ) from the input channel corresponding to  $B_{k_2}$  in (5.28) ; hence  $K(s)$  cannot be made proper by any auxiliary feedback  $Q(s)$  in (5.29). Indications are, however, that the only viable  $Q(s)$  (viable in the sense that it produces a well-posed system matrix for  $K(s)$  at least) is one which may be parameterized in terms of  $\epsilon$  with a system matrix which is itself not well-posed in the limit as  $\epsilon \rightarrow 0$ . It is not clear how this can be constructed in general.

There is, however, a simpler alternative which we will discuss, somewhat briefly, in the next section.

## 5.5 A New Embedding for the Singular Problem

In this section we propose a new embedding for the singular  $H^\infty$  problem; this embedding allows us to obtain reduced order proper solutions for all singular problems.

Suppose first that  $D_{11} = 0$ ,  $D_{22} = 0$  and that  $D_{21}$  is surjective. We have already discussed the alleviation of the first two restrictions; we will shortly describe how the final constraint can be accommodated. Finally, we assume that  $G_{12}$  and  $G_{21}$  have no finite  $C^{0e}$  zeros.

If  $C_1 B_{2_2}$  is injective then Corollary 5.1 provides us with a solution to the problem. If, however,  $C_1 B_{2_2}$  is column rank deficient ( in which case  $M_{G_{12}}$  will have infinite zeros of minimum order 2) then we can proceed as follows: Choose a matrix  $C$  of

the same column dimension as  $C_1$  such that  $\begin{bmatrix} C_1 \\ C \end{bmatrix} B_{2_2}$  is injective (this can be done by using the singular value decomposition, for example [10]). Then by construction  $\hat{C}_\epsilon B_{2_2}$  is also injective, where

$$\hat{C}_\epsilon := \begin{bmatrix} C_1 \\ \epsilon C \end{bmatrix} \quad (5.112)$$

Now consider the new embedding (see Fig. 5.4: Note that, for simplicity, the  $D_{21}$  block has been omitted)

$$\tilde{G}_1(\epsilon, s) = \begin{bmatrix} \tilde{G}_{1_{11}}(\epsilon, s) & \tilde{G}_{1_{12}}(\epsilon, s) \\ \tilde{G}_{1_{21}}(\epsilon, s) & \tilde{G}_{1_{22}}(\epsilon, s) \end{bmatrix} \stackrel{s}{=} \left[ \begin{array}{c|ccc} -sI + A & B_1 & \vdots & B_2 \\ \hline C_1 & 0 & \vdots & 0 \\ \epsilon C & 0 & \vdots & 0 \\ \dots & \dots & \vdots & \dots \\ C_2 & D_{21} & \vdots & 0 \end{array} \right] \quad (5.113)$$

Note that the injectivity of  $C B_{2_2}$  guarantees that  $\tilde{G}_{1_{12}}(\epsilon, s)$  has zeros at infinity of order no greater than 1.

**Lemma 5.7** *There exists a controller  $K(s)$  which solves the suboptimal  $H^\infty$  problem for the plant  $G(s)$  iff there exists an  $\epsilon^* > 0$  for which a solution to the suboptimal  $H^\infty$  problem for the plant  $\tilde{G}_1(\epsilon^*, s)$  exists. Furthermore, if such an  $\epsilon^*$  exists, it holds that a solution to the problem exists for  $\tilde{G}_1(\epsilon, s) \forall \epsilon \in [0, \epsilon^*]$ .*

*Proof:* The proof follows the same line of argument given in the proof to Lemma 5.4. Suppose that there exists a compensator  $K(\epsilon^*, s)$  which solves the suboptimal

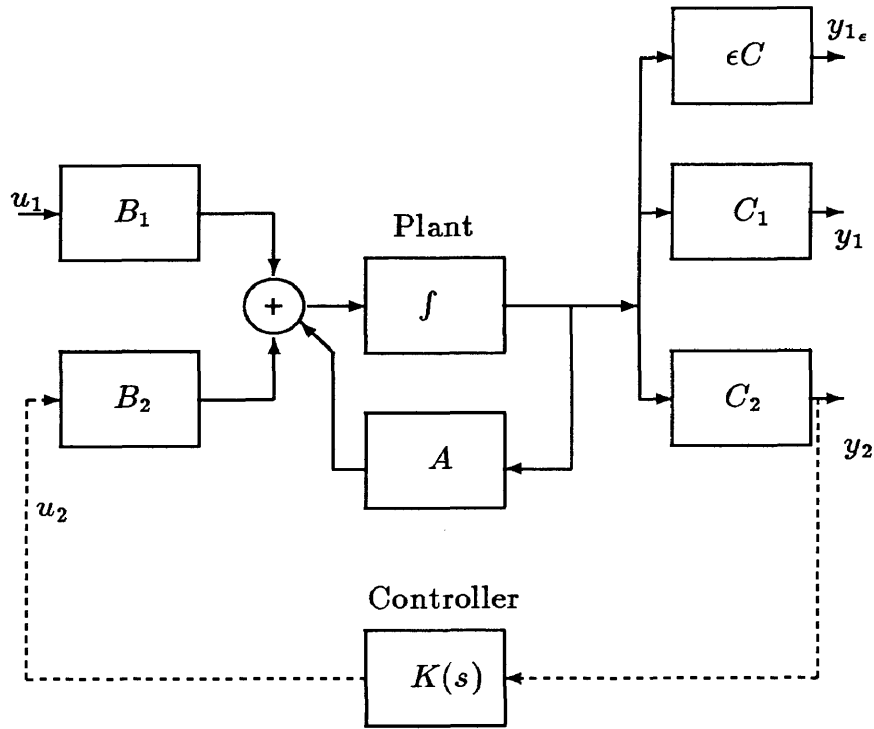


Figure 5.4: Another embedding of the plant  $G(s)$

problem for  $\tilde{G}_1(\epsilon^*, s)$  for some  $\epsilon^* > 0$ . To prove sufficiency, it suffices to let the controller for each  $\epsilon \in [0, \epsilon^*]$  to be given by  $K(\epsilon, s) = K(s) := K(\epsilon^*, s)$ . Then  $\forall \epsilon$  we have

$$T_{\hat{y}_1 u_1}(\epsilon, s) = \begin{bmatrix} T_{y_1 u_1}(s) \\ \epsilon(\hat{G}_{11}(s) + \hat{G}_{12}T_{22}(s)G_{21}(s)) \end{bmatrix} \quad (5.114)$$

where

$$\hat{G}_{11}(s) = C(sI - A)^{-1}B_1 \quad (5.115)$$

$$\hat{G}_{12}(s) = C(sI - A)^{-1}B_2 \quad (5.116)$$

$$T_{y_1 u_1}(s) = \mathcal{F}_l(G(s), K(s)) \quad (5.117)$$

and, as before,

$$\tilde{y}_1 := \begin{bmatrix} y_1 \\ y_{1e} \end{bmatrix} \quad (5.118)$$

$$T_{22}(s) := K(s)(I - G_{22}(s)K(s))^{-1} \quad (5.119)$$

Since each partition in (5.114) is a stable transfer function at  $\epsilon = \epsilon^*$  it follows that  $T_{\tilde{y}_1 u_1}(\epsilon, s) \in RH^\infty \forall \epsilon$  and, in particular,  $\forall \epsilon \in [0, \epsilon^*]$ . By simple matrix manipulations, it follows that  $\forall \epsilon \in [0, \epsilon^*] \|T_{\tilde{y}_1 u_1}(\epsilon, s)\|_\infty \leq \|T_{\tilde{y}_1 u_1}(\epsilon^*, s)\|_\infty < 1$ . In particular, we have that  $T_{y_1 u_1}(s) \in RH^\infty$ ,  $\|T_{y_1 u_1}(s)\|_\infty \leq \|T_{\tilde{y}_1 u_1}(\epsilon^*, s)\|_\infty < 1$ .

For necessity, assume that the controller  $K_0(s)$  solves the problem for the plant  $G(s)$ . Then we have that  $\|T_{y_1 u_1}\|_\infty = 1 - \delta$ ,  $1 > \delta > 0$ . Apply the controller  $K_0(s)$  and consider the closed loop system for increasing  $\epsilon$ . Since  $K_0(s)$  internally stabilizes  $\tilde{G}_1(0, s) \Rightarrow T_{y_1 u_1}$ ,  $T_{22}$ ,  $T_{22}G_{21}(s)$  and  $G_{12}T_{22}$  are all in  $RH^\infty$ ; moreover these matrices do not depend on  $\epsilon \Rightarrow T_{\tilde{y}_1 u_1} \in RH^\infty \forall \epsilon$ . Hence we have that  $k := \|\hat{G}_{11} + \hat{G}_{12}T_{22}G_{21}\|_\infty < \infty$ . Moreover,

$$\|T_{\tilde{y}_1 u_1}(\epsilon, s)\|_\infty = \left\| \begin{bmatrix} T_{y_1 u_1} \\ \epsilon(\hat{G}_{11} + \hat{G}_{12}T_{22}G_{21}) \end{bmatrix} \right\|_\infty \quad (5.120)$$

$$\leq \|T_{y_1 u_1}\|_\infty + \epsilon k \quad (5.121)$$

Now note that for  $\epsilon^* := \frac{\delta}{2k}$   $\|T_{\tilde{y}_1 u_1}(\epsilon, s)\|_\infty \leq 1 - \delta + \frac{\delta}{2} = 1 - \frac{\delta}{2} < 1 \forall \epsilon \in [0, \epsilon^*]$ .

□

Thus, as for our original embedding, this method provides us with a method for solving singular  $H^\infty$  problems. There is, however, an added advantage to using this new embedding as described in the following lemma:

**Lemma 5.8** *Assume that there is  $\epsilon^*$  for which there exists a solution to the  $H^\infty$  problem for  $\tilde{G}_1(\epsilon^*, s)$  with  $\lambda_{\max}(Q_0 P_0) < 1$ . Then there exists a controller of order at most  $n - r_{2_2}$  which solves the  $H^\infty$  problem for  $G(s)$ .*

*Proof:* Follows from the fact that the infinite zeros of  $\tilde{G}_{1,2}(\epsilon, s)$  are, by construction, of order unity, at most, and the application of Corollary 5.1 and Lemma 5.7.

□

*Comment:* Lemma 5.8 and Corollary 5.1 provide us a means of obtaining reduced order solutions to all singular  $H^\infty$  problems.

*Comment:* It should be noted that the  $\mathcal{C}^{0e}$  structure of singular problems will generically be simple, i. e. the  $\mathcal{C}^{0e}$  zeros which pertain to such problems will almost always be of order unity. This easily follows from the known sensitivity of Jordan or Jordan-like structures (for example, Weierstrass and Kronecker Canonical structures) associated with zeros of order 2 or higher [18, 12]. The embedding procedure above clarifies this for the case of infinite  $\mathcal{C}^{0e}$  zeros: for any  $\epsilon > 0$   $\tilde{G}_1(\epsilon, s)$  always has a simple  $\mathcal{C}^{0e}$  structure. In view of this, the result stated in Corollary 5.1 is therefore made much more significant.

*Comment:* Finally, we note that the constraint on  $D_{21}$  may be eliminated by employing a combination of both embeddings discussed in this chapter, i. e.

$$\tilde{G}_2(\epsilon, s) \stackrel{s}{=} \left[ \begin{array}{c|cc} -sI + A & \tilde{B}_1 & B_2 \\ \hline C_1 & 0 & 0 \\ \epsilon C & 0 & 0 \\ \dots & \dots & \dots \\ C_2 & \tilde{D}_{21} & 0 \end{array} \right] \quad (5.122)$$

where  $\tilde{B}_1$  and  $\tilde{D}_{21}$  are as defined in (5.35).

Since the results described above can be dualized to the case where  $D_{21}$ , say, is rank deficient it is seen that the embedding procedures described here allow us to obtain solutions to singular problems of order  $n - \min(\eta_{12}, \eta_{21})$  at most, where  $\eta_{12}$  and  $\eta_{21}$  are the respective nullities of  $D_{12}$  and  $D_{21}$ .

## 5.6 Some Examples

In what follows we apply our results to a few interesting examples.

Corollary 5.1 indicates that when  $u$  has dimension  $r = n$  the  $H^\infty$  compensator will be a constant gain. This is shown more clearly in the next example.

*Example 5.1:* If in Corollary 5.1 we have  $\text{rank } B_2 = n$ , all other assumptions re-

maining unchanged then we obtain  $\Xi_{12} = \begin{bmatrix} 0 \\ 0 \\ I_n \end{bmatrix}$  and

$$K_0(s) \stackrel{s}{=} \left[ \begin{array}{c|c} X_{21}^T B_2 & U_{21}^T \\ \hline I & 0 \end{array} \right] \quad (5.123)$$

which is a static compensator

$$\left. \begin{aligned} K_0(s) = K &= -B_2^{-1} X_{21}^{-T} U_{21}^T \\ &= B_2^{-1} (B_1 D_{21}^T + Q_0 C_2^T) (D_{21} D_{21}^T)^{-1} \end{aligned} \right\} \quad (5.124)$$

Where  $Q_0$  is as given in (5.98). This solution is implementable by output injection and may be compared to the result in [67].

The next example illustrates the fact that, in some cases, the limiting compensator can be improper.

*Example 5.2 (Francis [16, page 64]:* Here we have

$$G(s) = \left[ \begin{array}{c|c} \frac{1}{s+1} & \frac{1}{(s+1)^2} \\ \hline 1 & 0 \end{array} \right] \stackrel{s}{=} \left[ \begin{array}{cc|cc} -s-1 & 1 & 1 & 0 \\ 0 & -s-1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad (5.125)$$

Here  $D_{12} = 0$  so we require the infinite zero structure of  $C_1(sI - A)^{-1}B_2$ ; this

transfer function has all of its zeros at infinity. We therefore obtain  $\Xi_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$ .

$W_{21}(0, s)$  has  $2n$  finite zeros with  $\Lambda_{21} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$  and  $\Xi_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$ . It then

follows that the central controller is given by  $K_0(s) = -(s+1)$  which is easily shown to achieve the  $H^\infty$  optimum (see also [16])  $T_{y_1 u_1} \equiv 0$ .

*Example 5.3:* Take

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} \\ \frac{s+2}{s+1} & \frac{s+2}{(s+1)^2} \end{bmatrix} \stackrel{s}{=} \left[ \begin{array}{cc|cc} -s-1 & 1 & 1 & 0 \\ 0 & -s-1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right]. \quad (5.126)$$

Here  $W_{12}(0, s)$  has no finite zeros and we obtain  $\Xi_{12}$  and  $\Xi_{21}$  as in Example 5.2. In this case, however,  $\Lambda_{21} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$  and the central controller is

$$K_0(s) \stackrel{s}{=} \left[ \begin{array}{cc|c} 1 & -s & 0 \\ 0 & 0 & -1 \\ \hline -1 & -1 & 0 \end{array} \right] \quad (5.127)$$

By Lemma 4.3 the controller in (5.127) is not well-posed.



Let us now attempt to solve the problem for this plant using the embedding technique illustrated in Fig. 5.4. We take  $\epsilon = 1$  and  $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$  to get

$$\tilde{G}_1(1, s) \stackrel{s}{=} \left[ \begin{array}{cc|cc} -s-1 & 1 & 1 & 0 \\ 0 & -s-1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 0 \end{array} \right] \quad (5.128)$$

Solving the pertinent eigenproblems, we see that  $W_{12}(s)$  now has a finite eigenvalue at  $s = -\sqrt{3}$  and that  $\Lambda_{21}$  and  $\Xi_{21}$  are as before; moreover

$$\Xi_{12} = \begin{bmatrix} -(\sqrt{3}+1) & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1-\sqrt{3} & 1 \end{bmatrix}, \quad H_{12} = \begin{bmatrix} -\sqrt{3}-1 & 0 \\ 1 & 1 \end{bmatrix} \quad (5.129)$$

Note that  $H_{12}$  is nonsingular and that  $\lambda_{\max}(P_0Q_0) = 0 < 1$ . Substituting for  $K_0(s)$  we obtain

$$K_0(s) \stackrel{s}{=} \left[ \begin{array}{cc|c} -s-1 & -1 & 0 \\ (\sqrt{3}+1)(s+2) & -1 & 1 \\ \hline \sqrt{3}-1 & -1 & 0 \end{array} \right] \quad (5.130)$$

which has transfer function  $K_0(s) = K_0 := -\frac{1}{\sqrt{3}+2}$ . For this compensator,  $T_{y_1u_1}(s) = \frac{s+1}{s^2+(2-K_0)s+1-2K_0}$  which is stable (the poles are at  $s = -1.13 \pm 0.5j$ ) and has infinity norm  $\frac{1}{1-2K_0} < 1$  (see Fig 5.5).

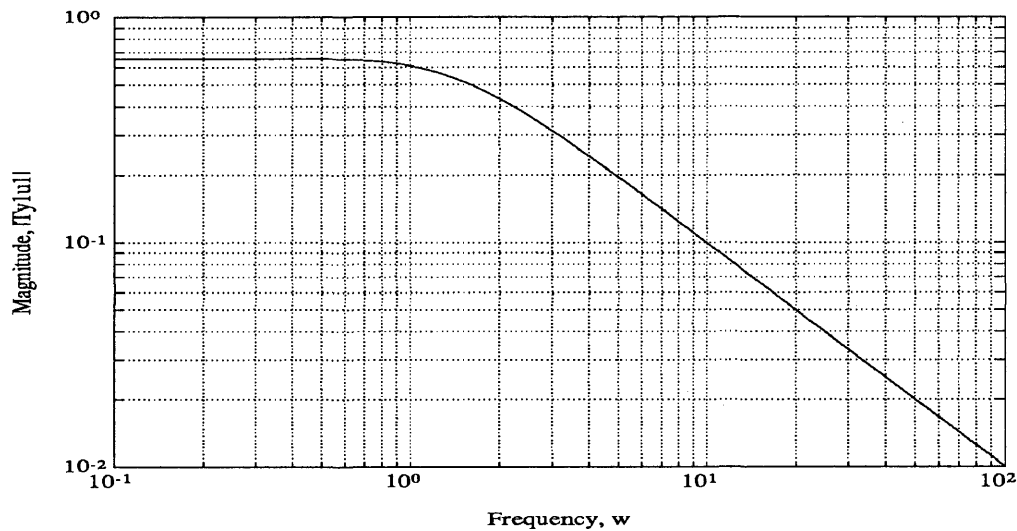


Figure 5.5: Magnitude Plot of  $T_{y_1 u_1}$  for Example 5.3

## 5.7 Summary and Conclusion

In this chapter we have proposed a procedure for solving the singular  $H^\infty$  problem. Theorem 5.1 shows that in such cases the infinite zero structure of  $M_{G_{12}}(s)$  and  $M_{G_{21}}(s)$  play a critical role in the required solution.

The replacement of the usual Riccati equations in [14, 35] by two generalized eigenproblems involving the pencils in (5.5) and (5.24) has proven to be pivotal to the analysis carried out. We iterate that, in our view, this method of solution would tend to be less sensitive to numerical problems. We therefore expect the procedure to be numerically well behaved for  $\epsilon$  arbitrarily small; by comparison, Petersen has noted the problems with the determination of the solution to the Riccati equation which occur using the orthodox approach. We also expect improved numerical

performance in the regular case when the matrices  $D'_{12}D_{12}$  or  $D_{21}D'_{21}$  are poorly conditioned for inversion.

The perturbation procedures described in Sections 5.3 and 5.5 provide straightforward methods of solving the singular problem by solving a nearby regular one. By comparison, the analysis in [57, 56], although very enlightening, requires that we perform input/output, state and feedback transformations, solve the Riccati equations for a regular subproblem and finally derive the required controller by determining the solution to a pertinent almost disturbance decoupling problem. The latter is essentially a high gain feedback problem which must, in general, be solved by an iterative process in any case. On this basis we claim that it is much more effective to directly apply the iterative scheme of Sections 5.3, 5.5. We note, however, that the technique used in [57, 56] allows one to decouple the interaction between  $\gamma$  and  $\epsilon$  in the double-iteration process.

We have shown in Corollary 5.1 that a reduction in the order of the central controller is possible when the order of the infinite zeros of  $M_{G_{12}}(s)$  or  $M_{G_{21}}(s)$  is unity. This can be shown to result from the replacement of the observer (or dual observer) structure in the central controller by a reduced order observer (or dual observer). The reader can consult the proof of the main result in [14] for verification of this. Of course, this is similar to what obtains in the LQG problem (see Chapter 4 and [41]). The main point, though, is that in this case, it is quite beneficial to let  $\epsilon \rightarrow 0$ . Moreover, the reduced order structure is generated by the same procedure

used to synthesize the usual  $n$ -th order controller; no separate formulae or procedures are required as is usually specified in the literature [41] for the LQG case.

Finally, we have derived an embedding which allows us to exploit the result described in Corollary to obtain reduced order solutions to all singular problems for which the only  $\mathcal{C}^{0e}$  zeros are at infinity. These solutions have been shown to have irreducible state-space realizations of maximum order  $n - \min(\eta_{12}, \eta_{21})$  where  $\eta_{12}$  and  $\eta_{21}$  are the respective nullities of  $D_{12}$  and  $D_{21}$ .

## Chapter 6

### Zero Canceling Compensators for FDLTI

#### Systems

#### 6.1 Introduction

In what follows we consider first the problem of obtaining a compensator  $U_\infty(s)$  for the proper  $m \times r$  transfer function

$$G(s) = C(sI - A)^{-1}B + D \quad (6.1)$$

such that  $G_c(s) = G(s)U_\infty(s)$  is non-strictly proper and  $G_c(s)$  has the same poles as  $G(s)$ . We shall call such a compensator an infinite zero compensator (IZC) of  $G(s)$  and the procedure which we shall describe will, of course, result in  $U_\infty(s)$  being polynomial. We subsequently show how the theory developed can be easily extended to case where  $G(s)$  has zeros on  $\mathcal{C}^0 := \{j\omega, \omega \in \mathfrak{R}\}$ . A compensator,  $U_0(s)$  which corrects for the zeros of  $G(s)$  on  $\mathcal{C}^0$  will be termed a  $\mathcal{C}^0$  zero compensator ( $\mathcal{C}^0\text{ZC}$ ). Clearly, we can compensate for zeros on  $\mathcal{C}^{0e} := \{j\omega, \omega \in \mathfrak{R}\}$ , the extended

imaginary axis by combining both procedures; in keeping with the terminology already introduced, we shall call the pertinent compensator a  $\mathcal{C}^{0e}$  zero compensator ( $\mathcal{C}^{0e}ZC$ ).

We make the following assumptions

**A6.1**  $G(s)$  is a tall matrix, i. e. it has as least as many rows as columns.

**A6.2**  $G(s)$  is non-degenerate, i. e. it has full normal rank.

**A6.3**  $\{C, A\}$  is detectable.

As indicated in Chapter 1, the ultimate purpose for studying this problem is to employ these compensators in solving singular control problems. Indeed, it should be clear that if  $U_{0e}(s)$  is a  $\mathcal{C}^{0e}ZC$  for the transfer function  $G(s)$ , then  $U_{0e}^{-1}(s)$ , contains the entire  $\mathcal{C}^{0e}$  structure of  $G(s)$ . It is essentially the knowledge of this structure which we hope to exploit in solving these problems. In Chapter 7 we will apply this compensation approach to solve the singular inner/outer factorization problem. While we will not be addressing singular  $H^2/H^\infty$  we do include, in this chapter, an example which indicates how the theory can be applied.

## 6.2 Transfer Function Approach to IZC Design

It is a simple matter to determine an IZC from (2.2). Specifically, rewrite (2.2) as

$$G(s)R(s) = L^{-1}(s)D(s) \quad (6.2)$$

and let

$$D_c(s) := \text{diag}\{\phi_1(s), \dots, \phi_r(s)\} \quad (6.3)$$

where  $\phi_i(s)$  is any polynomial of degree  $k_i$ , i. e.

$$\phi_i(s) := \phi_{i_{k_i}} s^{k_i} + \dots + \phi_{i_0}, \quad \phi_{i_{k_i}} \neq 0 \quad (6.4)$$

Since  $\phi_i(s)$  has, by construction,  $k_i$  poles at infinity, it follows that the  $k_i$  infinite zeros of  $d_i(s)$  are cancelled when the product  $d_i\phi_i(s)$  is formed. Hence  $DD_c(s)$  is biproper and, since  $L(s)$  is also biproper  $\Rightarrow L^{-1}DD_c(s)$  is biproper and therefore has no zeros at infinity. It therefore follows from (6.2) that  $RD_c(s)$  is an IZC of  $G(s)$ .

We will now show how to obtain a strictly polynomial IZC of  $G(s)$ ; this would result in a compensated transfer function which has the same poles as  $G(s)$  (save, perhaps for those poles of  $G(s)$  which are cancelled by zeros of the compensator).

Define

$$\tilde{U}_\infty(s) := \text{pol}(RD_c(s)) \quad (6.5)$$

where  $\text{pol}(M(s))$  denotes the polynomial part of the function  $M(s)$ . Define also

$$\tilde{U}_f(s) := RD_c(s) - \tilde{U}_\infty(s) \quad (6.6)$$

Then we have the following theorem:

**Theorem 6.1** *Let  $\tilde{U}_f(s)$  and  $\tilde{U}_\infty(s)$  be defined as above. Then*

(i)  $\tilde{U}_f(s)$  is strictly proper.

(ii)  $G\tilde{U}_\infty(s)$  is proper and has no infinite zeros, i. e.  $\tilde{U}_\infty(s)$  is an IZC of  $G(s)$ .

(iii)  $\tilde{U}_\infty(s)$  has the descriptor representation

$$\tilde{U}_\infty(s) \stackrel{s}{=} \left[ \begin{array}{c|c} \text{diag}\{s\Lambda_1 - I_{k_1+1}, \dots, s\Lambda_r - I_{k_r+1}\} & B_c \\ \hline \Xi_{21} \quad \dots \quad \Xi_{2r} & 0 \end{array} \right] \quad (6.7)$$

where

$$B_c := \text{diag}\{B_{c_1}, \dots, B_{c_r}\} \quad (6.8)$$

$$B_{c_i} = \left[ \begin{array}{cccc} \phi_{i0} & \phi_{i1} & \dots & \phi_{ik_i} \end{array} \right]^T \quad (6.9)$$

$$\Xi_{2i} := \left[ \begin{array}{ccc} R_{i0} & \dots & R_{ik_i} \end{array} \right] \quad (6.10)$$

where  $R_{ij}$  is the coefficient of  $s^{-j}$  in the  $i$ -th column of  $R(s)$  and  $\Lambda_i$  is the  $(k_i + 1) \times (k_i + 1)$  nilpotent matrix

$$\Lambda_i = \left[ \begin{array}{c|c} 0_{k_i \times 1} & I_{k_i} \\ \hline 0 & 0_{1 \times k_i} \end{array} \right] \quad (6.11)$$

*Proof:* Part (i) follows directly from the definition of  $\tilde{U}_f(s)$ . It follows that  $G\tilde{U}_f(s)$  is also strictly proper. Observe now that

$$G\tilde{U}_\infty(s) = L^{-1}DD_c(s) - G\tilde{U}_f(s). \quad (6.12)$$

We have already shown that  $L^{-1}DD_c(s)$  is proper and has no infinite zeros; it therefore follows from (6.12) that  $G\tilde{U}_\infty(s)$  also has this property. This is best seen by assuming that the transfer functions in (6.12) have the following state-space forms:

$$\left. \begin{array}{l} G\tilde{U}_\infty(s) = D_1 + C_1(sI - A_1)^{-1}B_1 \\ G\tilde{U}_f(s) = D_2 + C_2(sI - A_2)^{-1}B_2 \\ L^{-1}DD_c(s) = D_3 + C_3(sI - A_3)^{-1}B_3 \end{array} \right\} \quad (6.13)$$



and noting that  $G\tilde{U}_f(s)$  strictly proper  $\Rightarrow D_2 = 0$ ; thus  $D_1 = D_3$  and is therefore injective since  $L^{-1}DD_c$  has no infinite zeros. The result (ii) then follows.

Finally, we note that the  $i$ -th column of  $\tilde{U}_\infty(s)$  is

$$\tilde{U}_{\infty_i}(s) = \text{pol}(R_i(s)\phi_i(s)) \quad (6.14)$$

$$= \text{pol} \left( \begin{array}{c} \left[ \begin{array}{cccc} R_{i0} & R_{i1} & \cdots & R_{ik_i} \end{array} \right] \begin{bmatrix} 1 \\ s^{-1} \\ \vdots \\ s^{-k_i} \end{bmatrix} \left[ \begin{array}{cccc} 1 & s & \cdots & s^{k_i} \end{array} \right] B_{c_i} \end{array} \right) \quad (6.15)$$

$$= \text{pol} \left( \Xi_{2i} \begin{bmatrix} 1 & s & \cdots & s^{k_i-1} & s^{k_i} \\ s^{-1} & 1 & \ddots & & s^{k_i-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s^{-k_i+1} & \ddots & \ddots & \ddots & s \\ s^{-k_i} & s^{-k_i+1} & \cdots & s^{-1} & 1 \end{bmatrix} B_{c_i} \right) \quad (6.16)$$

$$= \Xi_{2i} \begin{bmatrix} 1 & s & & s^{k_i-1} & s^{k_i} \\ & \ddots & \ddots & & s^{k_i-1} \\ & \circ & \ddots & \ddots & \\ & & & \ddots & s \\ & & & & 1 \end{bmatrix} B_{c_i} \quad (6.17)$$

$$= \Xi_{2i}(-s\Lambda_i + I_{k_i+1})^{-1}B_{c_i} \quad (6.18)$$

which establishes (6.7). □

*Comment:* Note that by our previous discussions, it suffices to replace  $R_{i_j}$  in the theorem by a grade  $j + 1$  eigenvector  $f_{i_j}$  corresponding to the  $i$ -th zero of  $G(s)$  at infinity.

To conclude this section, we present a corollary to Theorem 6.1 which expands on the property of infinite zero eigenvectors described in (2.6); we shall then present a proof to Lemma 2.2.

**Corollary 6.1** *Define*

$$M_i := \begin{bmatrix} CA^{k_i-1}B & \dots & CB & D \end{bmatrix} \begin{pmatrix} R_{i_0} \\ \vdots \\ \vdots \\ R_{i_{k_i}} \end{pmatrix} \quad (6.19)$$

where the  $R_{i_j}$  are as defined in Theorem 6.1. Then

$$M := \begin{bmatrix} M_1 & \dots & M_r \end{bmatrix} \quad (6.20)$$

is injective.

*Proof:* Choose

$$\phi_i(s) = s^{k_i} \quad (6.21)$$

in (6.4). Then from (6.15)

$$\tilde{U}_{\infty_i}(s) = R_{i_0}s^{k_i} + \dots + R_{i_{k_i}} \quad (6.22)$$

Now observe that

$$G\tilde{U}_{\infty_i}(s) = \left( D + \frac{CB}{s} + \frac{CAB}{s^2} + \dots \right) \tilde{U}_{\infty_i}(s) \quad (6.23)$$

from which we see that the constant term in the expansion of the LHS of (6.23) is equal to  $M_i$  as defined in (6.19). It follows that  $M = D_1$ , where  $D_1$  is given in (6.13), and is therefore injective.  $\square$

*Proof to Lemma 2.2:* Sufficiency follows directly from Lemma 2.1. For necessity we note the following: If  $\{f_{i_0}, \dots, f_{i_{k_i}}\}$ ,  $i \in \underline{r}$  are a set of eigenvector chains at infinity then, by definition,  $f_{i_j} = R_{i_j}$ ,  $i \in \underline{r}$ ,  $j \in \underline{k_i} \cup \{0\}$  where  $R_{i_j}$  is, as before, the coefficient of  $s^{-j}$  in the  $i$ -th column of some biproper  $R(s)$  which effects the decomposition (2.2). Since  $R(s)$  is biproper, it follows that  $R_0 = \begin{bmatrix} R_{i_0} & \dots & R_{r_0} \end{bmatrix}$  is nonsingular; this implies (2.7). The fact that (2.6) holds has, as noted before, been shown in [63] while (2.8) is implied by Corollary 6.1.

□

### 6.3 Transfer Matrix Approach to IZC Design

The design procedure described in the previous section requires a preliminary decomposition of the form (2.2). Since the procedure requires us to operate directly on the system transfer function it is expected that this method would be somewhat lacking in the area of numerical reliability. In this section we describe a much more feasible method for obtaining an IZC of a specified transfer function,  $G(s)$ , by applying unitary transformations to the corresponding state-space system matrix. To this end we note that Lemma 2.7 implies the following: Define

$$\Xi_2 := \begin{bmatrix} \Xi_{2_1} & \dots & \Xi_{2_r} \end{bmatrix} \quad (6.24)$$

where  $\Xi_{2_i}$  is as defined in (6.10). Let  $n_\infty = \sum_{i=1}^r k_i$ . Then there exists an  $n \times (n_\infty + r)$  matrix  $\Xi_1$  such that  $\begin{bmatrix} \Xi_1 \\ \Xi_2 \end{bmatrix}$  spans the infinite frequency eigenspace of  $P(s)$ , i. e.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Xi_1 \\ \Xi_2 \end{bmatrix} \Lambda = \begin{bmatrix} \Xi_1 \\ 0 \end{bmatrix} \quad (6.25)$$

where  $\Lambda$  is the nilpotent matrix

$$\Lambda = \text{diag}\{\Lambda_1, \dots, \Lambda_r\} \quad (6.26)$$

and  $\Lambda_i$  is as defined in (6.11). Thus it is seen that the IZC  $\tilde{U}_\infty(s)$  in (6.7) can be obtained by determining the infinite eigenvalue structure of  $P(s)$ .

Because we anticipate numerical problems in determining the form in (6.25) we shall proceed to derive a basis-free representation of  $\tilde{U}_\infty(s)$ . The next lemma is useful in this regard. As before we assume that  $G(s)$  has the Rosenbrock system matrix

$$M_G(s) = \left[ \begin{array}{c|c} -sI + A & B \\ \hline C & D \end{array} \right] \quad (6.27)$$

**Lemma 6.1** *There exist unitary transformations  $S, T$  such that*

$$SM_G(s)T = \left[ \begin{array}{c|c} -sE_{\eta_f} + A_{\eta_f} & 0 \\ \hline * & -sE_\infty + A_\infty \end{array} \right] \quad (6.28)$$

where the regular pencil  $-sE_\infty + A_\infty$  contains all the infinite elementary divisors of  $M_G(s)$ .

*Proof:* The proof is constructive and follows directly from Algorithm 4.1 of [11].

Note that, by assumption A2,  $M_G(s)$  has no column null structure.  $\square$

Let us now assume a partitioning of  $S, T$  in conformity with the partitioning of  $M_G(s)$  in (6.27) and the LHS of (6.28), i. e.  $S = [S_{ij\infty}]$ ,  $T = [T_{ij\infty}]$ ,  $i, j \in \underline{2}$ . Define

$$U_\infty(s) \stackrel{s}{=} M_{U_\infty}(s) = \left[ \begin{array}{c|c} -sE_\infty + A_\infty & B_\infty \\ \hline T_{22\infty} & D_\infty \end{array} \right] \quad (6.29)$$

where  $B_\infty, D_\infty$  are chosen so that  $U_\infty(s)$  is regular and has all its zeros at prescribed locations in the complex plane (the relevant details are given below).

*Comment:* Observe that the pencil  $-sE_\infty + A_\infty$  has the dimensions of  $\Lambda$ , i. e.  $(n_\infty + r) \times (n_\infty + r)$ , with  $\text{rank } E_\infty = n_\infty$ .

**Lemma 6.2** *With the indicated partitioning of  $S, T$  the following hold*

(i) *The matrix  $A_\infty^{-1}E_\infty$  is nilpotent and*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_{12\infty} \\ T_{22\infty} \end{bmatrix} A_\infty^{-1}E_\infty = \begin{bmatrix} T_{12\infty} \\ 0 \end{bmatrix} \quad (6.30)$$

(ii)  $\begin{bmatrix} T_{12\infty} \\ T_{22\infty} \end{bmatrix}$  *spans the infinite eigenspace of  $M_G(s)$ .*

(iii) *The triple  $\{T_{22\infty}, A_\infty, E_\infty\}$  is observable.*

(iv) *Assume that*

$$-sE_\infty + A_\infty = \left[ \begin{array}{c|c} -sE_{11} + A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right] \quad (6.31)$$

where  $E_{11} \in \mathfrak{R}^{n_\infty \times n_\infty}$  is nonsingular for  $E_\infty \neq 0$ . Partition  $T_{12\infty}$  and  $T_{22\infty}$  in accordance with the RHS of (6.31), i. e.

$$T_{12\infty} = \begin{bmatrix} T_{12\infty_1} & T_{12\infty_2} \end{bmatrix}, \quad T_{22\infty} = \begin{bmatrix} T_{22\infty_1} & T_{22\infty_2} \end{bmatrix} \quad (6.32)$$

Then  $T_{12\infty_2} = 0$  and  $T_{22\infty_2} \in \mathfrak{R}^{r \times r}$  is nonsingular.

*Comment:* The form (6.31) may be obtained by application of the singular value decomposition (SVD) to the pencil  $-sE_\infty + A_\infty$ .

*Proof:* Note first that  $A_\infty$  must indeed be nonsingular and  $A_\infty^{-1}E_\infty$  nilpotent since all the zeros of  $-sE_\infty + A_\infty$  are at infinity. From (6.28) we have that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_{12_\infty} \\ T_{22_\infty} \end{bmatrix} = \begin{bmatrix} S_{21_\infty}^T A_\infty \\ S_{22_\infty}^T A_\infty \end{bmatrix} \quad (6.33)$$

and

$$\begin{bmatrix} T_{12_\infty} \\ 0 \end{bmatrix} = \begin{bmatrix} S_{21_\infty}^T E_\infty \\ S_{22_\infty}^T E_\infty \end{bmatrix} \quad (6.34)$$

Rightmultiply (6.33) by  $A_\infty^{-1}E_\infty$  and substitute for the RHS using (6.34) to prove

(i). Part (ii) follows by comparing (6.30) with (2.22) and the fact that the pencil  $-sE_\infty + A_\infty$  contains all the infinite zeros of  $M_G(s)$  (Lemma 6.1).

For (iii) we first note that since the pencil  $-sE_\infty + A_\infty$  has only infinite zeros, the pencil  $\begin{bmatrix} -sE_\infty + A_\infty \\ T_{22_\infty} \end{bmatrix}$  is injective  $\forall s \in \mathcal{C}$ . It suffices, therefore, to show that this pencil has no infinite zeros. We will prove this by contradiction.

Assume that  $\begin{bmatrix} -sE_\infty + A_\infty \\ T_{22_\infty} \end{bmatrix}$  has at least one zero at infinity; then by Lemma 2.3 there is at least one eigenvector chain at infinity of minimum length 2. From (2.26) this implies the existence of non-zero vectors  $v_1$  and  $v_2$  satisfying

$$\begin{bmatrix} -sE_\infty + A_\infty \\ T_{22_\infty} \end{bmatrix} v_1 = \begin{bmatrix} E_\infty v_2 \\ 0 \end{bmatrix} \quad (6.35)$$

Note that from (2.26)  $E_\infty v_1 = 0$ . From (6.28)

$$S \begin{bmatrix} -sI + A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_{12_\infty} \\ T_{22_\infty} \end{bmatrix} = \begin{bmatrix} 0 \\ -E_\infty + A_\infty \end{bmatrix} \quad (6.36)$$

⇒

$$S \begin{bmatrix} -sI + A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_{12\infty} \\ T_{22\infty} \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ E_\infty v_2 \end{bmatrix} \quad (6.37)$$

⇒

$$\begin{bmatrix} -sI + A \\ C \end{bmatrix} T_{12\infty} v_1 = \begin{bmatrix} S_{21}^T \\ S_{22}^T \end{bmatrix} E_\infty v_2 = \begin{bmatrix} T_{12\infty} v_2 \\ 0 \end{bmatrix} \quad (6.38)$$

This last equation indicates that the observability pencil  $\begin{bmatrix} -sI + A \\ C \end{bmatrix}$  has at least one zero at infinity corresponding to eigenvector  $T_{12\infty} v_1$ ; this, however, contradicts the fact that the upper block of this pencil can only have finite zeros<sup>1</sup>. Hence the assumption that  $\begin{bmatrix} -sE_\infty + A_\infty \\ T_{22\infty} \end{bmatrix}$  is column rank deficient must be false which proves the result.

Finally, for part (iv) note that for  $E_\infty = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}$  substitution in (6.34) implies that  $T_{12\infty_2} = 0$ . Further substitution in (6.33) ⇒

$$\begin{bmatrix} B \\ D \end{bmatrix} T_{22\infty_2} = \begin{bmatrix} S_{21}^T \\ S_{22}^T \end{bmatrix} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \quad (6.39)$$

Since  $S$  is orthogonal,  $A_\infty$  is nonsingular and  $G(s)$  is nondegenerate it follows that the matrices  $\begin{bmatrix} S_{21}^T \\ S_{22}^T \end{bmatrix}$ ,  $\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$  and  $\begin{bmatrix} B \\ D \end{bmatrix}$  all have full column rank. It therefore follows that  $T_{22\infty_2}$  is nonsingular (its dimensions can be verified from (6.31, 6.32)).

□

We are now in a position to state the following

---

<sup>1</sup>Equivalently, we note that the state-space representation of any finite dimensional linear time-invariant system is always observable/controllable at infinity. There are several other contradictions evident in this equation.

**Theorem 6.2**  $U_\infty(s)$  is an input IZC of  $G(s)$ . Moreover the compensated transfer matrix  $G_{c_\infty}(s) := G(s)U_\infty(s)$  is given by

$$G_{c_\infty}(s) \stackrel{s}{=} \left[ \begin{array}{c|c} -sI + A & -S_{21\infty}^T B_\infty + BD_\infty \\ \hline C & -S_{22\infty}^T B_\infty + DD_\infty \end{array} \right] \quad (6.40)$$

*Proof:* We note that  $G_{c_\infty}(s)$  has the cascade realization

$$G_{c_\infty}(s) \stackrel{s}{=} M_{G_{c_\infty}}(s) := \left[ \begin{array}{cc|c} -sE_\infty + A_\infty & 0 & B_\infty \\ BT_{22\infty} & -sI + A & BD_\infty \\ \hline DT_{22\infty} & C & DD_\infty \end{array} \right] \quad (6.41)$$

Now perform the following sse operations on  $G_{c_\infty}(s)$ :

- (i) column1 = column1 + column2  $\times T_{12\infty}$
- (ii) row 2 = row 2  $- S_{21\infty}^T \times$  row 1
- (iii) row 3 = row 3  $- S_{22\infty}^T \times$  row 1

This gives

$$G_{c_\infty}(s) \stackrel{s}{=} \tilde{P}_{G_{c_\infty}}(s) := \left[ \begin{array}{cc|c} -sE_\infty + A_\infty & 0 & B_\infty \\ -s(T_{12\infty} - S_{21\infty}^T E_\infty) + AT_{12\infty} + BT_{22\infty} - S_{21\infty}^T A_\infty & -sI + A & -S_{21\infty}^T B_\infty + BD_\infty \\ \hline sS_{22\infty}^T E_\infty + CT_{12\infty} + DT_{22\infty} - S_{22\infty}^T A_\infty & C & -S_{22\infty}^T B_\infty + DD_\infty \end{array} \right] \quad (6.42)$$

By substitution from (6.33,6.34) the (2,1) and (3,1) entries of the LHS are zero  $\Rightarrow$  (6.40) is a (not necessarily minimal) realization of  $G_{c_\infty}(s)$ . Note that this also implies that the (1,1) block in (6.43) is unobservable.

Now we show that  $G_{c_\infty}(s)$  has no zeros at infinity. To do this we first use the cascade realization (6.41) and note that  $U_\infty(s)$  has no infinite zeros by design. Hence



the only infinite zeros of  $M_{G_{c_\infty}}(s)$  are those which may be attributed to  $G(s)$ . But these are precisely the zeros of the pencil  $-sE_\infty + A_\infty$  which forms the (1,1) block in (6.43) and which has been shown to be unobservable. It follows directly that  $G_{c_\infty}(s)$  has no infinite zeros.  $\square$

We will now discuss how  $B_\infty$  may be obtained so that  $U_\infty(s)$  is regular with a prescribed set of zeros in  $\mathcal{C}$ . We assume without loss of generality that  $-sE_\infty + A_\infty$  is in the form (6.31). It will also be assumed that  $B_\infty$  is suitably partitioned to give

$$U_\infty(s) \stackrel{s}{=} M_{U_\infty}(s) = \left[ \begin{array}{cc|c} -sE_{11} + A_{11} & A_{12} & B_{\infty_1} \\ A_{21} & A_{22} & B_{\infty_2} \\ \hline T_{22_{\infty_1}} & T_{22_{\infty_2}} & D_\infty \end{array} \right] \quad (6.44)$$

where  $T_{22_\infty}$  is partitioned as in (6.32). Recall that from Lemma 6.2  $T_{22_{\infty_2}}$  is nonsingular and set  $D_\infty = 0$ . Define

$$\hat{A}_{11} := A_{11} - A_{12}T_{22_{\infty_2}}^{-1}T_{22_{\infty_1}} \quad (6.45)$$

$$\hat{C} := A_{21} - A_{22}T_{22_{\infty_2}}^{-1}T_{22_{\infty_1}} \quad (6.46)$$

We are now in a position to state the following lemma which will be useful in determining a suitable value for  $B_\infty$ .

**Lemma 6.3**  $U_\infty(s)$  is regular with  $n_\infty$  zeros in  $\mathcal{C}$  iff  $B_{\infty_2} \in \mathfrak{R}^{r \times r}$  is nonsingular.

Moreover, the zeros of  $U_\infty(s)$  are the zeros of the pencil  $-sE_{11} + \hat{A}_{11} - B_{\infty_1}B_{\infty_2}^{-1}\hat{C}$  which can be placed arbitrarily in  $\mathcal{C}$  by suitable choice of  $B_{\infty_1} \in \mathfrak{R}^{n_\infty \times r}$ .

*Proof:* By using (2.26) it can be easily verified that an arbitrary pencil  $N(s) = \left[ \begin{array}{cc} -sE + A & B \\ C & D \end{array} \right]$ , where  $E$  is nonsingular, has no eigenvector chains at infinity of

length  $k_i > 1$  ( and therefore has only infinite zeros of order 0) iff  $D$  is injective. It then follows that the pencil  $M_{U_\infty}(s)$  in (6.44) has no infinite zeros iff  $\begin{bmatrix} A_{22} & B_{\infty_2} \\ T_{22\infty_2} & 0 \end{bmatrix}$  is nonsingular  $\Leftrightarrow B_{\infty_2}$  must also be nonsingular. The rest follows by noting that the zeros of  $U_\infty(s)$  are then given by the roots of

$$0 = \det(M_{U_\infty}(s)) = \det(-sE_{11} + \hat{A}_{11} - B_{\infty_1}B_{\infty_2}^{-1}\hat{C}). \quad (6.47)$$

By Lemma 6.2  $\{T_{22\infty_2}, A_\infty, E_\infty\}$  is observable  $\Rightarrow \{\hat{C}, \hat{A}_{11}, E_{11}\}$  is also observable indicating that the zeros of  $U_\infty(s)$  may be arbitrarily placed in  $\mathcal{C}$ .  $\square$

*Comment:* An obvious choice for  $B_{\infty_2}$  is  $B_{\infty_2} = I_r$ .

h

## 6.4 Zero Compensation on $\mathcal{C}^0$

In this section we show that the procedure developed above can be modified to design a  $\mathcal{C}^0$ ZC. We begin with the following lemma:

**Lemma 6.4** *There exist unitary transformations  $S, T$  such that*

$$SM_G(s)T = \left[ \begin{array}{c|c} -sE_{rest} + A_{rest} & 0 \\ \hline * & -sE_j + A_j \end{array} \right] \quad (6.48)$$

where the regular pencil  $-sE_j + A_j$  contains only the  $\mathcal{C}^0$  elementary divisors of  $M_G(s)$ .

*Proof:* As in Lemma 6.1 the proof is constructive; in this case it follows by application of Algorithm 4.5 of [11] and the fact that  $M_G(s)$  has, by assumption, no column null structure. □

Again we partition the unitary matrices  $S, T$  in conformity with the LHS of (6.48) and state the following (see also Lemma 6.2)

**Lemma 6.5** *With the indicated partitionings of  $S, T$  the following hold*

(i) *The  $C^0$  zeros of  $M_G(s)$  are the eigenvalues of the matrix  $E_j^{-1}A_j$  and*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_{12,j} \\ T_{22,j} \end{bmatrix} = \begin{bmatrix} T_{12,j} \\ 0 \end{bmatrix} E_j^{-1}A_j \quad (6.49)$$

(ii)  $\begin{bmatrix} T_{12,j} \\ 0 \end{bmatrix}$  *spans the finite eigenspace of  $M_G(s)$  corresponding to its  $C^0$  zeros.*

(iii) *The triple  $\{T_{22,j}, A_j, E_j\}$  is observable.*

(iv)  $S_{22,j} = 0$ .

*Proof:* The proofs for (i) and (ii) are similar to that for Lemma 6.2(i) except that now we employ the fact that  $E_j$  is nonsingular (if there is a zero at the origin

then  $A_j$  may be singular). For part (iii) suppose that  $\exists$  a vector  $v \neq 0$  such that

$$\begin{bmatrix} -j\omega E_j + A_j \\ T_{22,j} \end{bmatrix} v = 0. \text{ Then from (6.48) and the fact that the columns of } \begin{bmatrix} T_{12,j} \\ T_{22,j} \end{bmatrix}$$

are unitary it follows that  $\exists$  a vector  $w := T_{12,j}v \neq 0$  such that  $\begin{bmatrix} -j\omega I + A \\ C \end{bmatrix} w = 0$ .

This however contradicts the detectability assumption A6.3. Hence the result (iii) must hold.

For (iv) note that from (6.48),  $S_{22}^T E_j = 0$ ; the result then follows from the fact that  $E_j$  is nonsingular. □

Now we define

$$U_j(s) \stackrel{s}{=} M_{U_j}(s) := \left[ \begin{array}{c|c} -sE_j + A_j & B_j \\ \hline T_{22j} & D_j \end{array} \right] \quad (6.50)$$

and state the following adaptation of Theorem 6.2:

**Theorem 6.3**  $U_j(s)$  is a  $C^0$  ZC of  $G(s)$ . Furthermore the compensated transfer function  $G_{c_j} := G(s)U_j(s)$  is given by

$$G_{c_j}(s) \stackrel{s}{=} \left[ \begin{array}{c|c} -sI + A & -S_{21}^T B_j + B D_j \\ \hline C & D D_j \end{array} \right] \quad (6.51)$$

*Proof:* The proof proceeds along lines similar to that given for Theorem 6.2 and is therefore omitted. □

The zero placement problem in this case is simpler than for the infinite zeros case. We first note that  $D_j$  must be nonsingular to guarantee that no  $j\omega$ -axis zeros are shifted to infinity. Without loss of generality, we therefore assume that  $D_j = I_r$ .

**Lemma 6.6** Suppose  $E_j \in \mathfrak{R}^{n_j \times n_j}$ . Then the  $n_j$  zeros of  $U_j(s)$  are the zeros of the pencil  $-sE_j + A_j - B_j T_{22j}$ . These may be arbitrarily placed in  $\mathcal{C}$  by suitable choice of the matrix  $B_j \in \mathfrak{R}^{n_j \times r}$ .

*Proof:* Clearly the zeros of  $U_j(s)$  are the zeros of the pencil  $-sE_j + A_j - B_j T_{22j}$ . By Lemma 6.5  $\{T_{22j}, A_j, E_j\}$  is observable  $\Rightarrow$  the zeros of  $G(s)$  may be arbitrarily relocated. □

## 6.5 Compensation Algorithms

In this section we state our first main result in the form of three algorithms which utilize the theory discussed above.

### Algorithm 6.1: Infinite zero compensation

**Input:** State space parameters of  $G(s) = C(sI - A)^{-1}B + D$ .

**Step 4** Form  $M_G(s) = \left[ \begin{array}{c|c} -sI + A & B \\ \hline C & D \end{array} \right]$ .

**Step 5** Apply Algorithm 4.1 of [11] to  $M_G(s)$  to obtain the unitary transformations

$S, T$  which yield the lower block-triangular form (6.28) i. e.  $\left[ \begin{array}{c|c} * & 0 \\ \hline * & -sE_\infty + A_\infty \end{array} \right]$ .

Partition  $S, T$  as described in the discussions following Lemma 1.

**Step 6** Determine the SVD of  $E_\infty$ , i. e.  $E_\infty = U\Sigma V$ . Set  $E_\infty = \Sigma$ ,  $A_\infty = U^T A_\infty V$ ,  $S_{21_\infty} = U^T S_{21_\infty}$ ,  $S_{22_\infty} = U^T S_{22_\infty}$ ,  $T_{12_\infty} = T_{12_\infty} V$ ,  $T_{22_\infty} = T_{22_\infty} V$ . This transforms  $-sE_\infty + A_\infty$  into the form (6.31).

**Step 7** Determine the partitions  $E_{11}$ ,  $A_{11}$ ,  $A_{12}$ ,  $A_{21}$ ,  $A_{22}$ ,  $T_{22_{\infty_1}}$  and  $T_{22_{\infty_2}}$  using (6.31) and calculate  $\hat{A}_{11}$  and  $\hat{C}$  from (6.45, 6.46).

**Step 8** Using any pole placement algorithm, determine  $B_{\infty_1} \in \mathfrak{R}^{n_\infty \times r}$  to place the  $n_\infty$  poles of the matrix  $E_{11}^{-1}(\hat{A}_{11} - B_{\infty_1}\hat{C})$  as desired (see Lemma 6.3). Set

$$B_\infty = \begin{bmatrix} B_{\infty_1} \\ I_r \end{bmatrix}.$$

**Output:**  $S_{21\infty}, S_{22\infty}, T_{12\infty}, T_{22\infty}, E_\infty, A_\infty, B_\infty$ . The IZC is a polynomial matrix given by  $U_\infty(s) = T_{22\infty}(sE_\infty - A_\infty)^{-1}B_\infty$ .

*Comment:* Note that the number ( $n_\infty = \text{rank } E_\infty$ ) of compensator zeros cannot be specified *a priori*.

**Algorithm 6.2: Zero compensation on the  $j\omega$ -axis**

**Input:** The state space parameters of  $G(s) = C(sI - A)^{-1}B + D$ .

**Step 1** Form  $M_G(s) = \left[ \begin{array}{c|c} -sI + A & B \\ \hline C & D \end{array} \right]$ .

**Step 2** Apply Algorithm 4.5 of [11] to obtain the lower block-triangular form

$$S_0 M_G(s) T_0 = \left[ \begin{array}{c|c} -sE_\eta + A_\eta & 0 \\ \hline * & -sE_f + A_f \end{array} \right] \quad (6.52)$$

where the regular pencil  $-sE_f + A_f$  contains only the finite elementary divisors of  $M_G(s)$  and  $S_0, T_0$  are unitary.

**Step 3** Use the QZ algorithm to determine the  $\mathcal{C}^0$  zeros of  $-sE_f + A_f$ . Denote these by  $j\omega_i, i = 1, 2, \dots, k$ .

**Step 4** For each  $\omega_i, i \in \underline{k}$  apply Algorithm 3.1 of [11] to  $-sE_f + A_f$  to get

$$S_k \cdots S_1 (-sE_f + A_f) T_1 \cdots T_k = \left[ \begin{array}{c|c} -sE_{f_{rest}} + A_{f_{rest}} & 0 \\ \hline * & -s\hat{E}_j + \hat{A}_j \end{array} \right] \quad (6.53)$$

where now  $-s\hat{E}_j + \hat{A}_j$  contains all finite elementary divisors of  $M_G(s)$  on  $\mathcal{C}^0$ .

**Step 5** Let  $\hat{S} = \text{diag}\{I, S_k S_{k-1} \cdots S_1\} S_0$ ,  $\hat{T} = T_0 \text{diag}\{I, T_1 T_2 \cdots T_k\}$ . Partition  $\hat{S}$ ,  $\hat{T}$  as described in the discussion following Lemma 4.

**Step 6** Obtain orthogonal matrices  $L$ ,  $R$  such that  $S_{21j} = L \hat{S}_{21}$  and  $\begin{bmatrix} T_{12j} \\ T_{22j} \end{bmatrix} =$

$\begin{bmatrix} \hat{T}_{12} \\ \hat{T}_{22} \end{bmatrix} R$  are real projections of the indicated partitions of  $\hat{S}$  and  $\hat{T}$  respectively.

**Step 7** Set  $E_j = L \hat{E}_j R$ ,  $A_j = L \hat{A}_j R$ . Using any pole placement algorithm, determine  $B_j$  to place the poles of the matrix  $E_j^{-1}(A_j - B_j T_{22j})$  as desired.

**Output:**  $E_j$ ,  $A_j$ ,  $B_j$ ,  $S_{21j}$ ,  $T_{12j}$ ,  $T_{22j}$ . The  $\mathcal{C}^0\text{ZC}$  is a proper rational matrix  $U_j(s) = T_{22j}(sE_j - A_j)^{-1}B_j$ .

*Comment:* It is preferable to extract the infinite frequency structure of  $M_G(s)$  prior to the application of the QZ algorithm [11] since the latter, if applied directly to  $M_G(s)$  could wrongly detect infinite zeros as (large) finite ones [11]. Step 6 is necessary since  $\hat{S}$ ,  $\hat{T}$  will, in general, be complex matrices;  $L$ ,  $R$  may be determined using Givens rotations [21].

Zero compensation on  $\mathcal{C}^{0e}$  may be achieved by combining the previous algorithms as described in Algorithm 6.3 which follows.

**Algorithm 6.3: Zero compensation on the extended  $j\omega$ -axis**

**Input:** The state space parameters for  $G(s) = C(sI - A)^{-1}B + D$ .

**Step 1** Form  $M_G(s) = \left[ \begin{array}{c|c} -sI + A & B \\ \hline C & D \end{array} \right]$ .

**Step 2** Determine the IZC for  $G(s)$  by Algorithm 6.1.

**Step 3** Determine the compensated transfer function  $G_{c_\infty}(s)$  as specified in (6.40).

**Step 4** Determine the  $\mathcal{C}^0$ ZC for  $G_{c_\infty}(s)$  using Algorithm 6.2.

**Output:**  $S_{21j}, S_{22j}, S_{12\infty}, S_{22\infty}, T_{12j}, T_{22j}, T_{12\infty}, T_{22\infty}, E_j, A_j, E_\infty, A_\infty, B_j, B_\infty$ .

$U_{0e}(s)$  is given by the cascade

$$U_{0e}(s) := U_\infty U_j(s) \stackrel{s}{=} \left[ \begin{array}{cc|c} -sE_j + A_j & 0 & B_j \\ B_\infty T_{22j} & -sE_\infty + A_\infty & B_\infty \\ \hline 0 & T_{22\infty} & 0 \end{array} \right] \quad (6.54)$$

and  $G_{c_{0e}}(s) := G(s)U_{0e}(s)$  satisfies

$$G_{c_{0e}}(s) \stackrel{s}{=} \left[ \begin{array}{c|c} -sI + A & \hat{B} \\ \hline C & \hat{D} \end{array} \right] \quad (6.55)$$

where

$$\hat{B} := -(S_{21\infty} B_\infty + S_{21j} B_j) \quad (6.56)$$

$$\hat{D} := -(S_{22\infty} B_\infty) \quad (6.57)$$

We now consider the application of the procedures developed to the following example (pg. 300, Brogan [4]):



Example 6.1

$$G(s) = \begin{bmatrix} \frac{s^2+2s+3}{(s+1)^2} & \frac{s^2+6s+3}{(s+1)^2} \\ \frac{1}{s+1} & -\frac{1}{s+1} \\ \frac{-s+2}{(s+1)^2} & \frac{7s+4}{(s+1)^2} \end{bmatrix} \quad (6.58)$$

for which the state-space parameters are

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 3 & -1 & 6 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (6.59)$$

By application of Algorithm 6.3 we determine that  $G(s)$  has 1 zero at infinity and none on the  $j\omega$ -axis. Moreover a suitable zero compensator is

$$U_{0e}(s) \stackrel{s}{=} \left[ \begin{array}{ccc|cc} -s - 0.6667 & 2.3094 & -1.1547 & -3.5078 & 0 \\ -2.6607 & 0.4607 & 0.4607 & 1 & 0 \\ -3.780 & -1.2060 & -1.2060 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right] \quad (6.60)$$

which has a zero at  $s = -10.0$ . For the compensated system (6.55) we have that

$$\hat{B} = \begin{bmatrix} 2.0880 & 0.5767 \\ 1.9939 & -0.2884 \\ -1.9939 & 0.2884 \end{bmatrix} \quad (6.61)$$

and

$$\hat{D} = \begin{bmatrix} -0.5234 & 0.6293 \\ 0.2058 & 0.0786 \\ -0.8233 & -0.3145 \end{bmatrix} \quad (6.62)$$

Note that the matrix  $\hat{D}$  has full column rank implying that the compensated transfer function has no zeros at infinity.

We conclude this section with an example which demonstrates how these compensators may be applied to  $H^\infty$  (and  $H^2$ ) problems:

*Example 6.2*

Let us consider the following example for which we wish to solve the  $H^\infty$  problem.

We take  $G(s)$  in Example 5.3, i. e.

$$G(s) = \left[ \begin{array}{c|c} \frac{1}{s+1} & \frac{1}{(s+1)^2} \\ \hline \frac{s+2}{s+1} & \frac{s+2}{(s+1)^2} \end{array} \right] \quad (6.63)$$

For  $K(s) = -\frac{\alpha}{1-\alpha} \frac{(s+1)^2}{s+2}$ ,  $\alpha \in [0, 1)$ , we have  $T_{zw} = \frac{1-\alpha}{s+1} \Rightarrow \|T_{zw}\|_\infty = 1 - \alpha < 1$ .

Moreover,  $\|T_{zw}\|_\infty$  may be made arbitrarily small but the infimum  $T_{zw} = 0$  cannot be achieved with this controller (requires an infinite controller gain). In fact, analysis of the equation

$$T_{zw} = G_{11} + G_{12}K(1 - G_{22}K)^{-1}G_{21} = 0 \quad (6.64)$$

leads us to the conclusion that no  $K(s)$  exists which makes  $\|T_{zw}\| = 0$ .

Let us now apply the theory developed to this example. We apply the IZC  $U(s) = (s + 1)^2$  in Block # 1 (Fig.6.1) to obtain

$$G_1(s) = \left[ \begin{array}{c|c} \frac{1}{s+1} & 1 \\ \hline \frac{s+2}{s+1} & s+2 \end{array} \right]. \quad (6.65)$$

Note that the (2,2) block is improper. We cancel the improper part of the relevant transfer function using Block # 2. This gives

$$G_2(s) = \left[ \begin{array}{c|c} \frac{1}{s+1} & 1 \\ \hline \frac{s+2}{s+1} & 0 \end{array} \right]. \quad (6.66)$$

Now we solve the  $H^\infty$  problem for  $G_2(s)$ . An obvious candidate for a  $H^\infty$  controller is

$$K_2(s) = -\frac{1}{s+2} \quad (6.67)$$

which, incidentally, is also the solution obtained using the standard procedure (Lemma 5.3) and which results in  $\mathcal{F}_l(G_2(s), K_2(s)) \equiv 0$ . It is immediately obvious from the figure, however, that  $K_1(s)$  which results from the feedback composition of  $K_2(s)$  and Block 2 is not well-defined. In fact, this interconnection results in a unity loop gain configuration. Of course, satisfactory solutions can be constructed as was shown above and it can be shown that a proper solution may be recovered by use of the contractive operator  $Q(s)$  in Theorem 5.3. For example,  $Q(s) = \frac{1}{s+2}$  yields  $K(s) = -\frac{s+1}{s+2}$  in this case. It is not clear how to choose  $Q(s)$  to obtain a proper solution in general.

The characteristics of the singular problem highlighted above give some indication of the level of difficulty involved in deriving a solution. In particular, they show, at least from a practical perspective, that it may at times be pointless to seek a solution to the singular optimal  $H^\infty$  problem, since the resulting solution may be improper and hence unrealizable. Moreover, the behavior of the system (6.63) shows that even if improper controllers were allowed, there are some serious issues to be considered before attempting to solve the problem directly, as opposed to solving

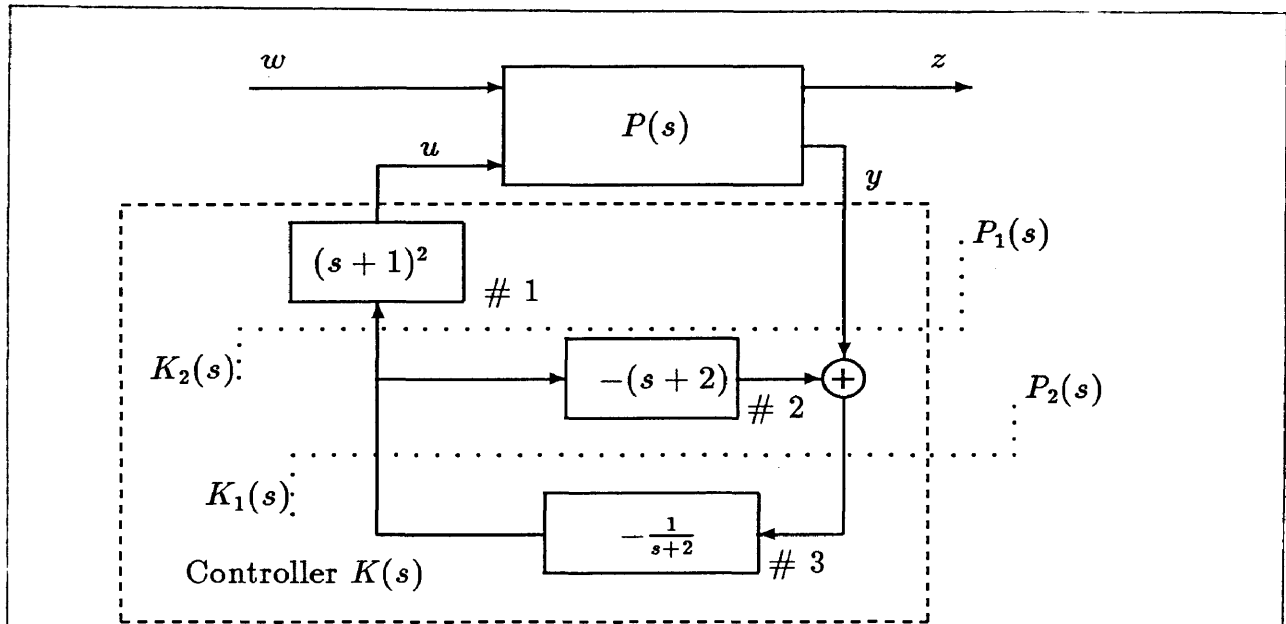


Figure 6.1:  $H^\infty$  Control using Infinite Zero Compensation

a nearby problem. In [57, 56] this issue, though not directly addressed, is made transparent through the role of the almost disturbance decoupling problem which the authors have shown to be embedded in the singular  $H^\infty$  problem. For cases such as these, we find that, except in certain situations, the limiting form of the cheap  $H^\infty$  compensator does not exist.

The example above shows that the compensation approach might be useful in giving us some insight into the issue of the existence of a solution to the cheap  $H^\infty$  problem. In particular, it would seem that the major factor here is the infinite zero structure of  $G_{22}(s)$ .

## 6.6 Summary and Conclusion

In the above, we have described simple procedures for obtaining a  $\mathcal{C}^{0e}ZC$  without recourse to the usual recursive methods of, say, [54]. As noted in Chapter 1, it is expected that such compensators would be useful in the solution of singular control problems. We believe that the algorithms presented allow for the derivation of the required compensator,  $U_{0e}(s)$ , in a numerically reliable fashion.

The starting point for our analysis has been the definition of the infinite zero eigenvectors of a transfer function  $G(s)$  as originally proposed in [63] and expanded upon in Chapter 2 (Definition 2.1).

The procedures described for determining  $B_\infty$  and  $B_j$  (Lemmas 6.3, 6.6) may be improved by resorting to descriptor pole placement techniques (see [2], for example). Also, some savings in computation may be achieved by performing the decomposition (6.48) on the reduced pencil  $-sE_{\eta f} + A_{\eta f}$  (6.28). This would also allow us to treat the design problem on  $\mathcal{C}^{0e}$  as a whole, rather than having to consider the point at infinity separately.

Finally, it is to be noted that the assumption A6.2 may be eliminated by appropriately modifying the design procedures. Specifically, degenerate transfer functions may be accommodated by first noting that the decomposition (2.2) now yields

$$D(s) = \begin{bmatrix} \text{diag}\{d_1(s), \dots, d_l(s), 0, 0, \dots, 0\} \\ 0_{(m-r) \times r} \end{bmatrix} \quad (6.68)$$

in (2.3) for a  $G(s)$  of rank  $l$ . To obtain an IZC we can use  $D_c(s)$  in (6.3) of the form

$$D_c(s) = \text{diag}(\phi_1(s), \dots, \phi_l(s), 1, \dots, 1) \quad (6.69)$$

with the unity elements corresponding to the null-space of  $G(s)$ . Once again the compensator may be constructed using the columns of  $R(s)$ ; note that the last  $r - l$  columns of  $R^{-1}(s)$  span the nullspace of  $G(s)$ . Since the degeneracy of  $G(s)$  is easily determined from the Kronecker canonical form of its system matrix, the required structural information may be obtained by applying the step algorithm [11]. Finally, note that we can only compensate or the “regular part” of  $G(s)$ ; the compensated transfer function always inherits the complete degeneracy structure of  $G(s)$ . The modification of our procedure to accommodate degenerate plants also allows us to obtain output compensators, say, for tall plants (or input compensators for “fat” plants).

## Chapter 7

### The Singular Inner/Outer Factorization Problem

The theory developed in the last chapter will now be applied to the problem of obtaining an inner-outer factorization (IOF) of a proper transfer function  $G(s) = C(sI - A)^{-1}B + D$  which is allowed to have zeros on  $\mathcal{C}^{0e}$ . It will be assumed that conditions A6.1-6.3 hold.

The background behind this particular problem has been discussed in Chapter 1.

#### 7.1 Singular IOF Algorithms

Using the zero canceling compensators of Chapter 6 we propose the following procedure for solving the singular IOF problem:

**Algorithm 7.1: Inner-outer factorization (I)**

**Input:** *Transfer function  $G(s)$ .*

**Step 1** Determine a  $C^{0e} ZC U_{0e}(s)$ , as described in Algorithm 6.3, with all of its zeros in the open LHP.

**Step 2** Determine an IOF of  $G_{c_{0e}}(s) := G(s)U_{0e}(s)$ , i. e.  $G_{c_{0e}}(s) = G_i \hat{G}_o(s)$ , using the MATLAB function `iofr` [6], for example.

**Step 3** From the above  $G(s) = G_i \hat{G}_o U_{0e}^{-1}(s)$ . This directly gives the inner factor as  $G_i(s) = G_{c_{0e}} \hat{G}_o^{-1}(s)$ ; the outer factor is  $G_o(s) = \hat{G}_o U_{0e}^{-1}(s)$ .

**Step 4** If required reduce  $G_o(s)$  to state-space form. This may be accomplished by application of the MATLAB function `des2ss` [6], for example. Alternatively,  $U_{0e}^{-1}(s)$  may be reduced to state space form prior to the formation of  $G_o(s)$ .

**Output:** Inner factor  $G_i(s)$ , outer factor  $G_o(s)$ .

Note that placing the zeros of  $U_{0e}(s)$  in the LHP allows us to adjoin its inverse to the outer factor. Also the factor  $G_i(s)$  may be determined from

$$G_i(s) = GG_o^{-1}(s) = G_{c_{0e}} \hat{G}_o^{-1}(s) \quad (7.1)$$

In order to give a more detailed procedure, we note that a realization for  $U_{0e}^{-1}(s)$  is given by

$$U_{0e}^{-1}(s) \stackrel{s}{=} \left[ \begin{array}{cc|c} -sE_j + A_j - B_j T_{22_j} & -B_j \hat{C} & -B_j A_{22} T_{22_{\infty_2}}^{-1} \\ 0 & -sE_{11} + \hat{A}_{11} - B_{\infty_1} \hat{C} & (A_{12} - B_{\infty_1} A_{22}) T_{22_{\infty_2}}^{-1} \\ \hline -T_{22_j} & -\hat{C} & -A_{22} T_{22_{\infty_2}}^{-1} \end{array} \right]. \quad (7.2)$$

The algorithm now proceeds as follows:



### Algorithm 7.2: Inner-outer factorization (II)

**Input:** State space parameters of  $G(s) = C(sI - A)^{-1}B + D$ .

**Step 1** Apply Algorithm 6.3 to obtain the compensated transfer function  $G_{c_{0e}}(s) = C(sI - A)^{-1}\hat{B} + \hat{D}$  of (6.55-6.57).

**Step 2** Set  $\underline{D} = \hat{D}\hat{D}^T$ .

**Step 3** Determine  $X$ , where  $X \geq 0$  solves the Riccati equation pertaining to the Hamiltonian matrix

$$H := \begin{bmatrix} A - \hat{B}\underline{D}^{-1}\hat{D}^T C & -\hat{B}\underline{D}^{-1}\hat{B}^T \\ -C^T C + C^T \hat{D}\underline{D}^{-1}\hat{D}^T C & -(A - \hat{B}\underline{D}^{-1}\hat{D}^T C)^T \end{bmatrix} \quad (7.3)$$

**Step 4** Form the inner and outer factors of  $G(s)$

$$G_o(s) \stackrel{s}{=} \left[ \begin{array}{ccc|c} -sE_j + A_j - B_j T_{22j} & -B_j \hat{C} & 0 & -B_j A_{22} T_{22\infty 2}^{-1} \\ 0 & -sE_{11} + \hat{A}_{11} - B_{\infty 1} \hat{C} & 0 & (A_{12} - B_{\infty 1} A_{22}) T_{22\infty 2}^{-1} \\ -\hat{B} T_{22j} & -\hat{B} \hat{C} & -sI + A & -\hat{B} A_{22} T_{22\infty 2}^{-1} \\ \hline -\underline{D}^{\frac{1}{2}} T_{22j} & -\underline{D}^{\frac{1}{2}} \hat{C} & \underline{D}^{-\frac{1}{2}} (\hat{D}^T C + \hat{B}^T X) & -\underline{D}^{\frac{1}{2}} A_{22} T_{22\infty 2}^{-1} \end{array} \right] \quad (7.4)$$

and

$$G_i(s) \stackrel{s}{=} \left[ \begin{array}{c|c} -sI + A - \hat{B}\underline{D}^{-1}(\hat{D}^T C + \hat{B}^T X) & \hat{B}\underline{D}^{-\frac{1}{2}} \\ \hline C - \hat{D}\underline{D}^{-1}(\hat{D}^T C + \hat{B}^T X) & \hat{D}\underline{D}^{-\frac{1}{2}} \end{array} \right] \quad (7.5)$$

**Output:** The inner factor,  $G_i(s)$ , and the outer factor,  $G_o(s)$ .

*Comment:* Steps 1,2,3 follow from [16]. The remaining step 4 embodies the general outline previously given. The formulae for  $G_o(s)$  and  $G_i(s)$  are obtained by simplifying  $G_c(s) = G_{c_{0e}}(s)\hat{G}_o^{-1}(s)$  and  $G_o(s) = \hat{G}_o(s)U_{0e}^{-1}(s)$  where,

$$\hat{G}_o(s) \stackrel{s}{=} \left[ \begin{array}{ccc|cc} -sI + A & & & \hat{B} & \\ \hline \underline{D}^{-\frac{1}{2}}(\hat{D}^T C + \hat{B}^T X) & & & \underline{D}^{\frac{1}{2}} & \end{array} \right] \quad (7.6)$$

is the outer factor of  $G_c(s)$ .

*Comment:* Algorithm 7.2 yields a state space realization for  $G_i(s)$  which is of lower dimension than that obtained from Algorithm 7.1.

We conclude this chapter with an example.

*Example* Take  $G(s)$  in (6.58) and apply Algorithm 7.1 to obtain the following inner-outer factorization:

$$G_i(s) \stackrel{s}{=} \left[ \begin{array}{ccc|cc} -s + 5.8684 & -2.5514 & 13.3171 & 2.1244 & 0.9113 \\ 7.1287 & -s - 4.1239 & 13.0048 & 1.9907 & -0.3181 \\ -7.1287 & 3.1239 & -s - 14.0048 & -1.9907 & 0.3181 \\ \hline -0.2471 & -0.3564 & -0.6035 & -0.4975 & 0.8675 \\ 0.6623 & 0.6430 & 1.3053 & 0.2109 & 0.1207 \\ 0.3509 & 0.4281 & 0.7789 & -0.8416 & -0.4826 \end{array} \right] \quad (7.7)$$

$$G_o(s) \stackrel{s}{=} \left[ \begin{array}{ccc|cc} -s - 10 & 0 & 0 & 0 & 3.9253 & 0.4612 \\ 5.7735 & -s - 1 & 1 & 0 & -0.2663 & -0.2663 \\ 5.1962 & 0 & -s - 1 & 0 & -1.2663 & -1.2663 \\ -5.1962 & 0 & 0 & -s - 1 & 1.2663 & 1.2663 \\ \hline 2.6394 & -3.4866 & 1.5971 & -6.4612 & -0.4975 & -0.4975 \\ 0.1825 & 0.5909 & 0.1740 & 0.4487 & 0.8675 & 0.8675 \end{array} \right] \quad (7.8)$$

By direct inspection, it can be seen that the direct feed-through matrix of the outer factor is rank deficient, verifying that this factor contains the infinite zero structure

of  $G(s)$ . Moreover the poles of  $G_o(s)$  are the poles of  $G(s)$  (these are all stable) and the single compensator zero at  $s = -10$ .

## Chapter 8

### Conclusion

#### 8.1 Thesis Summary

In this dissertation, we have

1. Comprehensively demonstrated the utility of the Generalized Eigenproblem formulation of the Riccati equation for solving both regular and singular control problems.

2. Used the Generalized Eigenproblem formulation to characterize the limiting trajectories for the cheap LQ and  $H^\infty$  problems with state feedback (Chapters 3, 5) and also to determine explicitly the limiting value of the solution to the relevant Riccati equations. The main foundation for the result is provided by a known result on the behavior of the invariant subspaces of a matrix valued function which is analytic in a scalar parameter,  $\epsilon$  (Lemma 3.6).

3. Used these results to formulate solutions to the cheap LQG and  $H^\infty$  problems with measurement feedback. The descriptor representation of FDLTI systems has facilitated the determination of improper solutions where necessary. We have also seen that in some cases there are no feedback solutions to these problems. On the other hand, we have considered cases where the solution to the cheap problem is not only proper, but also of order  $\delta < n$  and have proposed a perturbation scheme which enables one to achieve this property for all cheap LQG and  $H^\infty$  problems.

4. Formulated zero canceling compensators (Chapter 6). The design of infinite zero compensators, in particular, was proposed using the work of Verghese *et al.*[63] as a basis. The definitions and concepts which were described in this reference and which pertain to the eigenvector structure of FDLTI systems at infinity were expanded upon in Chapter 1.

The zero canceling compensators allow for the extraction of the “regular part” of the pertinent problem. Although the application of these compensators to the  $H^2$  and  $H^\infty$  problems has not been comprehensively discussed, we have illustrated the utility of the approach.

5. Solved the singular inner/outer factorization problem for nondegenerate transfer functions (Chapter 7) using the zero canceling compensators developed in Chapter 6.

## 8.2 Possibilities for Future Research

The following potential areas for research arising out of this work:

1. The application of the zero cancellation approach to the singular  $H^2$  and  $H^\infty$  problems. Work on this topic is currently underway [8]. Preliminary results have yielded the solutions described in Chapters 4, 5; this indicates, at least for the LQG problem, that the limiting solutions, if they exist, are indeed optimal. What is still not fully clear, however, is how one can pick an auxiliary compensator,  $Q(s)$ , which would fully exploit the singularity of the problem to yield a controller of order  $< n$  when the plant has at least second order zeros at infinity (Lemma 5.1).

We have already alluded to the role played by the infinite zero structure of  $G_{22}(s)$  in determining the existence of a solution to cheap LQG and  $H^\infty$  problems; this issue is, of course, also a major factor in singular problems and is currently being investigated in [8].

2. The application of the cancellation procedure to other singular problems; an example is the spectral factorization problem [16].

3. Finally we note that in this dissertation, we have focused on the problem of  $C^{0e}$  zeros and, in particular, we have concentrated on the effect of infinite zeros. Implicit in the usual  $H^2$  and  $H^\infty$  design procedures, however, is the restriction that the relevant system should have no poles at infinity, i. e. it cannot be improper. The  $H^\infty$  package in [6], for example, signals the user if such a situation occurs; adjustments must then be made in the control weights before computation can proceed. It

is instructive to consider how the situation involving an improper augmented plant may arise in practice. Consider again the mixed sensitivity problem in Fig. 1.1 where the design weights are invariably decided upon from performance (low frequency) specifications and plant uncertainty (high frequency) data. For simplicity, assume a single-input single-output plant  $G(s)$ . If the required roll-off, as specified by the roll-off rate of  $W_3$ , is too gentle and the original plant is itself strictly proper, then the augmented plant will inherit some of the infinite zero structure. On the other hand, if the roll-off rate is too great, i. e. if the required roll-off is greater than the plant roll-off, then the augmented plant will be improper. To the author's knowledge, the problem of  $G(s)$  being improper has not been fully treated in the literature. In Copeland and Safonov [9] this problem is considered for the case where the infinite poles are all either unobservable or uncontrollable, however a more detailed treatment is required. The work of Bender *et al.*[3] and others seems to be important in this regard.

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## Appendix A

### A.1 On the Non-Uniqueness of Eigenvectors at Infinity

Here we show how a new set of eigenvector chains at infinity can be generated by taking linear combinations of an existing set of chains. The result can be shown to hold for eigenvector chains of constant matrices. This lends further justification to the definition of eigenvector chains at infinity as proposed in Definition 1.1.

**Lemma A.1** *Suppose that  $G(s)$  has infinite zeros of respective orders  $k_i$ ,  $i \in \underline{r}$ . Also let  $\mathcal{N}_k$  be an index set containing all  $i \in \underline{r}$  for which the corresponding infinite zeros are of the same order  $k_i = k$  and  $\mathcal{N}$  be any (not necessarily complete) set of indices  $i$  for which the corresponding zeros at infinity are of minimum order  $k$ .*

Let  $z_{k,i}$  be a composite vector of the first  $k$  elements of the chain  $\{f_{i_0}, \dots, f_{i_{k_i}}\}$  which corresponds to the  $i$ -th infinite zero,  $i \in \mathcal{N}$  i. e.

$$z_{k,i} = \begin{bmatrix} f_{i_0} \\ \vdots \\ f_{i_{k-1}} \end{bmatrix} \quad (\text{A.1})$$

(i) Define

$$z = \sum_{i \in \mathcal{N}} \alpha_i z_{k,i} \quad (\text{A.2})$$

where the  $\alpha_i$ 's are arbitrary non-zero scalars and where  $\mathcal{N} \cap \mathcal{N}_k \neq \emptyset$ , i. e. the summation (A.2) includes at least one chain corresponding to a zero of order  $k$ . Then the elements of the composite vector  $\begin{pmatrix} z \\ w \end{pmatrix}$ ,  $w := \sum_{i \in \mathcal{N}} \alpha_i f_{i_k}$ , form an eigenvector chain at infinity corresponding to a zero of order  $k$ .

(ii) If in (i) above,  $\mathcal{N} \cap \mathcal{N}_k = \emptyset$ , then the elements of  $z$  form a partial chain corresponding to a zero of order  $\hat{k} := \min_{\mathcal{N}} \{k_i\}$ .

*Proof:* Part (i) can be ascertained by direct substitution in (2.6). For (ii) simply note that  $z_{\hat{k},i} = \begin{pmatrix} z_{k,i} \\ * \end{pmatrix}$ , take  $z = \sum_{\mathcal{N}} \alpha_i z_{\hat{k},i}$  and apply the result in (i). □

## A.2 Proof of Lemma 2.7

The following results are required to prove Lemma 2.7.

**Lemma A.1** Let  $S(s) = -sE + A$  be a regular matrix pencil with  $l$  eigenvalues at  $s = \lambda \in \mathcal{C} \cup \{\infty\}$  of respective orders  $q_i$ ,  $i \in \underline{l}$ , with corresponding eigenvector chains  $\{\xi_i^{(j)}\}_{i \in \underline{l}, j \in \underline{k_i}}$ , where

$$k_i = \begin{cases} q_i, & \lambda \in \mathcal{C} \\ q_i + 1, & \lambda = \infty \end{cases} \quad (\text{A.1})$$

Assume that the  $k_i$  are arranged in decreasing order i. e.

$$k_1 \geq k_2 \geq \dots \geq k_l.$$

Let  $z := \begin{pmatrix} z_1 \\ \vdots \\ z_l \end{pmatrix}$  be any vector  $\in \mathfrak{R}^l$  and define a new chain of vectors,  $\{y^{(j)}\}$  as follows:

$$y^{(j)} = \sum_{i \in \underline{l}} v_i^{(j-k_m+k_i)} z_i \quad (\text{A.2})$$

where

$$v_i^{(k)} = \begin{cases} \xi_i^{(k)}, & k > 0 \\ 0, & \text{otherwise} \end{cases} \quad (\text{A.3})$$

and  $i = m$  is the first index for which  $z_i \neq 0$ .

Then  $\{y^{(j)}\}$  is an eigenvector chain of  $S(s)$  corresponding to the eigenvalue  $s = \lambda$  of order  $q_m$ .

*Comment:* The new chain defined above is a linear combination of the original chains formed by alignment of the highest grade eigenvectors. Thus, the last element,  $y^{(k_m)}$ , of the chain  $\{y^{(j)}\}$  is always given by the implied linear combination of the highest grade eigenvectors corresponding to the eigenvalue  $s = \lambda$ . Similarly, the



next to last element is a linear combination of the next to highest grade eigenvectors, and so on. The shorter chains are padded with zero vectors to make their lengths compatible with that of the longest chain.

*Proof:* Without loss of generality, we assume that the pencil  $S(s)$  is in Kronecker canonical form. We will first prove the result for  $\lambda \in \mathcal{C}$ ; it therefore suffices to prove the result for a matrix

$$J = \text{diag}\{J_1, \dots, J_l\}$$

where each  $J_i$  is a  $k_i \times k_i$  Jordan matrix with eigenvalue  $s = \lambda$ .

First note that  $y^{(1)}$  is a grade 1 eigenvector corresponding to the eigenvalue at  $\lambda$  and that the chain  $\{y^{(j)}\}$  satisfies:

$$\left. \begin{aligned} (-\lambda I + J)y^{(1)} &= 0 \\ (-\lambda I + J)y^{(j+1)} &= y^{(j)}, \quad j = 2, \dots, k_m \end{aligned} \right\} \quad (\text{A.4})$$

Thus  $\{y^{(j)}\}$  is indeed an eigenvector chain corresponding to the eigenvalue  $\lambda$ ; since  $y^{(k_m)} \in \text{Ker}(J - \lambda I)^{k_m}$  this chain is of length  $k = k_m$  (see [32], for example). This proves the result for finite  $\lambda$ .

The proof for  $\lambda = \infty$  is obtained by repeating the above for the eigenvectors corresponding to the eigenvalue at  $\rho = 0$  for the pencil  $\hat{S}(\rho) = -E + \rho A$ .

□

**Lemma A.2** *Let*

$$\Xi_h := \begin{bmatrix} \xi_1^{(k_1+1)} & \dots & \xi_r^{(k_r+1)} \end{bmatrix} \in \mathfrak{R}^{(n+r) \times r} \quad (\text{A.5})$$

be a matrix of all eigenvectors of  $P(s)$  at infinity of the highest grade. Then the matrix

$$M := \begin{bmatrix} C & D \end{bmatrix} \Xi_h \quad (\text{A.6})$$

is injective.

*Proof:* Suppose that, contrary to the conclusion of the lemma,  $M$  is not injective.

Then  $\exists z = \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \in \mathfrak{R}^r$  such that

$$Mz = 0 \quad (\text{A.7})$$

Assume, without loss of generality, that  $k_1 \geq k_i$ ,  $i \in \underline{r}$ , and that  $z_1 \neq 0$ . Form a new vector chain  $\{y^{(j)}\}$  satisfying

$$y^{(j)} = \sum_{i \in \underline{r}} v_i^{(j-k_1+k_i)} z_i \quad (\text{A.8})$$

where

$$\left. \begin{aligned} v_i^{(k)} &= \xi_i^{(k)} & k > 0 \\ &= 0 & \text{otherwise} \end{aligned} \right\} \quad (\text{A.9})$$

By Lemma A.1 this new chain is an eigenvector chain of  $P(s)$  at infinity of length  $k = k_1$ . However, (A.7, 2.26) imply that it is an eigenvector chain of length  $k > k_1$  (from (2.26) it is easily verified that any vector  $v$  which is an eigenvector at infinity of less than the highest grade, satisfies  $Mv = 0$ , while those of the highest grade satisfy  $Mv \neq 0$ ). This is an obvious contradiction which indicates that the hypothesis that  $M$  is not injective is false; the result therefore follows.  $\square$

*Proof of Lemma 2.7:* Part (i) is proved in [60] by a simple extension of a similar result for finite zeros due to Rosenbrock [45].

For part (ii), note that  $\{u_{i_0}, \dots, u_{i_{k_i}}\}$ ,  $i \in \underline{r}$ , is an eigenvector chain of  $G(s)$  at infinity  $\Rightarrow$  from (2.6, 2.26) that (2.54) is satisfied and that the vectors  $\xi_i^{(k)}$  do indeed form chains of eigenvectors of  $P(s)$  at infinity.

On the other hand, suppose that the eigenvector chains  $\{\xi_i^{(1)}, \dots, \xi_i^{(k_i)}\}$ ,  $i \in \underline{r}$ , satisfy (2.54). Then since these vectors are linearly independent,

$$\text{rank} \begin{bmatrix} u_{1_0} & \dots & u_{l_0} \end{bmatrix} = r \quad (\text{A.10})$$

moreover, by (2.26), the chains  $\{u_{i_j}\}$  satisfy (2.6).

Define also  $M$  as in Lemma A.2 and note that its  $i$ -th column is given by

$$M_i = \begin{bmatrix} CA^{k_i-1}B & \dots & CB & D \end{bmatrix} \begin{bmatrix} u_{i_0} \\ \vdots \\ u_{i_{k_i}} \end{bmatrix} \quad (\text{A.11})$$

By Lemma A.2,  $M$  is injective  $\Rightarrow$  by Lemma 2.1 that  $\{u_{i_0}, \dots, u_{i_{k_i}}\}$ ,  $i \in \underline{r}$ , are eigenvector chains of  $G(s)$  at infinity.  $\square$ .

## Appendix B

### B.1 Optimal Cost

In what follows, we derive the cost associated with the transfer function  $T_{y_c w}(s)$  in (4.33).

First let

$$B_T := \begin{bmatrix} H'_{12} B_* \\ -(X'_{21} B_1 + U'_{21} D_{21}) \end{bmatrix} \quad (\text{B.1})$$

$$C_T := \begin{bmatrix} C_1 X_{12} + D_{12} U_{12} & -C_1 H_{21} \end{bmatrix} \quad (\text{B.2})$$

$$A_T := \begin{bmatrix} \Lambda_{12} & A_{T12} \\ 0 & \Lambda'_{21} \end{bmatrix} \quad (\text{B.3})$$

where

$$A_{T12} = H'_{12} (H_{21} \Lambda'_{21} - A H_{21}) \quad (\text{B.4})$$

Then note that from Theorem 1.53 [31] the limiting variance matrix  $Q := \begin{bmatrix} Q_{11} & Q_{12} \\ Q'_{12} & Q_{22} \end{bmatrix}$  in response to unit intensity white noise is the unique solution to

$$A_T Q + Q A'_T + B_T B'_T = 0 \quad (\text{B.5})$$

Moreover, the optimal cost  $J_*$  is given by

$$J_* = \text{Tr}(C'_T C_T Q) \quad (\text{B.6})$$

Expanding (B.5) we get

$$\Lambda_{12} Q_{11} + Q_{11} \Lambda'_{12} + A_{T_{12}} Q'_{12} + Q_{12} A'_{T_{12}} + H'_{12} B_* B'_* H_{12} = 0 \quad (\text{B.7})$$

$$\Lambda'_{21} Q_{22} + Q_{22} \Lambda_{21} + (X'_{21} B_1 + U'_{21} D_{21})(X'_{21} B_1 + U'_{21} D_{21})' = 0 \quad (\text{B.8})$$

$$\Lambda_{12} Q_{12} + Q_{12} \Lambda_{21} + A_{T_{12}} Q_2 - H'_{12} B_* (X'_{21} B_1 + U'_{21} D_{21})' = 0 \quad (\text{B.9})$$

We prove the result by a series of assertions:

**Assertion B.1** *The unique solution to  $Q_{22}$  in (B.8) is*

$$Q_{22} = X_{21} \Phi_{21} \quad (\text{B.10})$$

*Proof:* From

$$W_{21}(0, s) \begin{bmatrix} X_{21} \\ \Phi_{21} \\ U_{21} \end{bmatrix} = \begin{bmatrix} X_{21} \\ \Phi_{21} \\ 0 \end{bmatrix} (-sI + \Lambda_{21}) \quad (\text{B.11})$$

one obtains

$$\begin{aligned} (X'_{21} B_1 + U'_{21} D_{21})(X'_{21} B_1 + U'_{21} D_{21})' &= X'_{21} (X'_{21} B_1 + U'_{21} D_{21})' + \\ &\quad U'_{21} (X'_{21} B_1 + U'_{21} D_{21})' \end{aligned}$$

$$\begin{aligned}
&= -X'_{21}(A\Phi_{21} + \Phi_{21}\Lambda_{21}) - U'_{21}C_2\Phi_{21} \\
&= -(X'_{21}A + U'_{21}C_2)\Phi_{21} - X'_{21}\Phi_{21}\Lambda_{21} \\
&= -(\Lambda'_{21}X'_{21}\Phi_{21} + X'_{21}\Phi_{21}\Lambda_{21}) \quad (\text{B.12})
\end{aligned}$$

Hence (B.8) is satisfied by (B.10). By Lemma 1.5 of [31] the fact that  $\Lambda_{21}$  is strictly Hurwitz implies that this solution is unique. □

**Assertion B.2** *The unique solution  $Q_{12}$  in (B.9) is given by*

$$Q_{12} = 0 \quad (\text{B.13})$$

*Proof:* By direct substitution for  $A_{T_{12}}$  we have

$$\begin{aligned}
A_{T_{12}}Q_{12} - H'_{12}B_*(X'_{21}B_1 + U'_{21}D_{21})' &= \\
-H'_{12}(AH_{21}X'_{21}\Phi_{21} + B_1(X'_{21}B_1 + U'_{21}D_{21})' + H_{21}X'_{21}\Phi_{21}\Lambda_{21}) & \quad (\text{B.14})
\end{aligned}$$

$$= -H'_{12}(A(H_{21}X'_{21} - I)\Phi_{21} + (H_{21}X'_{21} - I)\Phi_{21}\Lambda_{21}) \quad (\text{B.15})$$

Note that in the last equation we have used the following relationship:

$$B_1(B'_1X_{21} + D'_{21}U_{21}) = -(A\Phi_{21} + \Phi_{21}\Lambda_{21}) \quad (\text{B.16})$$

By (3.48)  $(H_{21}X'_{21} - I)\Phi_{21} = 0$ . Hence the left hand side expression in (B.14) is zero. Substitution in (B.9) gives

$$\Lambda_{12}Q_{12} + Q_{12}\Lambda_{21} = 0 \quad (\text{B.17})$$

The result follows by application of Lemma 1.5 in [31] and the fact that  $\Lambda_{12}$ , and  $\Lambda_{21}$ , are both strictly Hurwitz.

□

Now expanding (B.6) we get

$$J_* = \text{Tr}[(C_1 X_{12_*} + D_{12} U_{12_*})' (C_1 X_{12_*} + D_{12} U_{12_*}) Q_{11}] + \text{Tr}[H'_{21_*} C'_1 C_1 H_{21_*} Q_{22}] \quad (\text{B.18})$$

The last term is

$$\begin{aligned} \text{Tr}[H'_{21_*} C'_1 C_1 H_{21_*} Q_{22}] &= \text{Tr}[C'_1 C_1 H_{21_*} Q_{22} H'_{21_*}] \\ &= \text{Tr}[C'_1 C_1 H_{21_*} X'_{21_*} \Phi_{21_*} H'_{21_*}] \\ &= \text{Tr}[C'_1 C_1 \Phi_{21_*} H'_{21_*}] \\ &= \text{Tr}[C'_1 C_1 P_{21_*}] \end{aligned} \quad (\text{B.19})$$

Use

$$\begin{aligned} (C_1 X_{12_*} + D_{12} U_{12_*})' (C_1 X_{12_*} + D_{12} U_{12_*}) &= -(X'_{12_*} (A \Phi_{12_*} + \Phi_{12_*} \Lambda_{12_*}) + U'_{12_*} B'_2 \Phi_{12_*}) \\ &= -(\Lambda'_{12_*} X'_{12_*} \Phi_{12_*} + X'_{12_*} \Phi_{12_*} \Lambda_{12_*}) \end{aligned} \quad (\text{B.20})$$

to get

$$\begin{aligned} \text{Tr}[(C_1 X_{12_*} + D_{12} U_{12_*})' (C_1 X_{12_*} + D_{12} U_{12_*}) Q_{11}] &= -\text{Tr}[\Lambda'_{12_*} X'_{12_*} \Phi_{12_*} Q_{11} + \\ &\quad X'_{12_*} \Phi_{12_*} \Lambda_{12_*} Q_{11}] \\ &= -\text{Tr}[(X_{12_*} \Phi_{12_*}) (\Lambda_{12_*} Q_{11} + Q_{11} \Lambda'_{12_*})] \end{aligned} \quad (\text{B.21})$$

From (B.9) and Assertion B.2 the first term on the right hand side of (B.18) is

$$\text{Tr}(X'_{12_*} \Phi_{12_*} H'_{12_*} B_* B'_* H_{12_*}) = \text{Tr} P_{12_*} B_* B'_* \quad (\text{B.22})$$

giving

$$J_* = \text{Tr}(P_{12_*} B_* B_*' + C_1' C_1 P_{21_*}). \quad (\text{B.23})$$



## Appendix C

### C.1 Limiting Transfer Function

**Assertion C.1** *The limit transfer function  $T_{y_{12}w}(s)$  is given by*

$$T_{y_{12}w}(s) = \mathcal{F}_l(G(s), F(s)) \quad (\text{C.1})$$

*Proof:* By direct substitution

$$\mathcal{F}_l(G(s), F(s)) = \left[ \begin{array}{cc|c} -sI + A & -B_2U_{12} & B_1 \\ U'_{21}C_2 & X'_{21}(-sI + A)X_{12} + X'_{21}B_2U_{12} + U'_{21}C_2X_{12} & U'_{21}D_{21} \\ \hline C_1 & -D_{12}U_{12} & 0 \end{array} \right] \quad (\text{C.2})$$

Now add  $X'_{21}$  by row 1 to row 2, subtract column 1 by  $X_{12}$  from column 2, multiply column 2 by -I and interchange columns 1 and 2 to get:

$$\mathcal{F}_l(G(s), F(s)) = \left[ \begin{array}{cc|c} (-sI + A)X_{12} + B_2U_{12} & -sI + A & B_1 \\ 0 & X'_{21}(-sI + A) + U'_{21}C_2 & U'_{21}D_{21} + X'_{21}B_1 \\ \hline C_1X_{12} + D_{12}U_{12} & C_1 & 0 \end{array} \right] \quad (\text{C.3})$$

Finally multiply row 1 by  $H'_{12}$  and column 2 by  $-H_{21}$  to obtain the matrix pencil listed in Table 4.1b and apply the same arguments as in the main body of the proof of the theorem to get the result. □