MATHEMATICS EDUCATION REFORM IN TRINIDAD AND TOBAGO: THE CASE OF REASONING AND PROOF IN SECONDARY SCHOOL

BY
ANDREW ANTHONY HUNTE

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Curriculum and Instruction
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2016

Urbana, Illinois

Doctoral Committee:

Associate Professor Gloriana González Rivera, Chair
Professor Cameron McCarthy
Professor Tami S. Martin
Associate Professor Monica McDermott

ABSTRACT

In Trinidad and Tobago, there have been substantive efforts to reform mathematics education. Through the implementation of new policies, the reformers have promoted changes in mathematics curriculum and instruction. A focus of the reform has been that of increasing opportunities for students to engage in reasoning and proving. However, little is known about how these policies affect the opportunities for reasoning and proof in the written curriculum, the teaching of proof, and students' learning. Furthermore, we are yet to know how the high-stake assessment measures interact with these new policies to impact the teaching and learning of proof.

In this dissertation, my overarching question asks: What are the implications of reform on the teaching and learning of secondary school mathematics in Trinidad and Tobago? To answer this question, I conducted three studies, which examined the content, teaching, and students' conceptions. All the studies are situated in the teaching reasoning and proof when introducing geometry concepts. In the first study, I conduct a curriculum analysis focused on examining the opportunities for reasoning and proof in the three recommended secondary school textbooks. In the second study, I conduct classroom observations of teachers' geometry instruction focusing on opportunities for engaging students in reasoning and proof. In the third study, I examine geometry students' conceptions of proof. The three studies are intended to provide an overview of the impact of reform on instructional issues in relation to the dynamics between teachers, student, and content (Cohen, Raudenbush, & Ball, 2003).

For the first study, I adapt a framework developed by Otten, Gilbertson, Males, and Clark (2014) to investigate the quality and quantity of the opportunities for students to engage in or reflect on reasoning and proof. The findings highlight some unique characteristics of the

recommended textbooks such as, (a) the promotion of the explanatory role of proof through the affordances of what I define as the Geometric Calculation with Number and Explanation (GCNE) exercises, (b) the necessary scaffolding of proof construction through activities and exercises promoting pattern identification, conjecturing, and developing of informal non-proof arguments, and (c) the varying advocacy for Geometry as an area in the curriculum where students can experience the work of real mathematicians and see the intellectual of proof in their mathematical experiences. All these characteristics align with the reformers' vision for the teaching and learning of reasoning and proof in secondary school mathematics.

In the second study, I examine the nature of the teaching of reasoning and proof in secondary school. I use classroom observations along with pre- and post-observations interview data of three teachers to determine (a) the classroom microculture (i.e., classroom mathematical practices and sociomathematical norms), (b) teachers' pedagogical decisions, and (c) teachers' use of the Caribbean Secondary Examination Certificate (CSEC) examination materials and textbooks. I also determine whether the teachers' instruction demonstrate the four characteristics of reform-based mathematics teaching (Hufferd-Ackles, Fuson, & Sherin, 2004). My analysis of classroom observations of the three teachers suggests that their instructional practices exhibit elements of reform-based instruction. These include teachers' use of open-ended and direct questions to solicit students' mathematical ideas and teachers' consideration of students as the source of mathematical ideas. Each teacher established sociomathematical norms that governed how and when a student can ask questions. In this case, questioning helped students articulate their ideas when responding to questions and clarifying their understanding of other's ideas when they posed a question. Teachers also established sociomathematical norms that outlined what counts as a valid proof and what counts as an acceptable answer during instruction. The

aforementioned norms supported the expectation that students must always provide explanations for their mathematical thinking, which is another characteristic of reform-based teaching. Teachers used group work and whole class discussions to offer opportunities for collaborative learning, which facilitated their creation of a social constructivist environment for learning reasoning and proof. Teachers used the reform-oriented curriculum materials to provide opportunities for construction of proofs. However, the textbooks and curriculum were limited in their support for proving some geometrical results. Overall, the teachers emphasized the making and testing of conjectures, which afforded students with authentic mathematical experiences that promoted the development of mathematical knowledge.

In the third study, I use the six principles of proof understanding (McCrone & Martin, 2009) to examine 21 students' conceptions of proof. I use semi-structured interviews to gather students' perspectives of (a) the roles of proof, (b) structure and generality of proof, and (c) the opportunities for proof in the curriculum materials. The findings indicate that the students considered proof as serving the roles of explanation, verification, systemization, and appreciation in mathematics. The latter role helps students see the value and purpose of the mathematical results they learn (a) for applications during problem solving and (b) within the axiomatic system of Geometry results. The aforementioned roles also help students see the intellectual need for reasoning and proof in their mathematical experiences. Students' talk suggests that, their teachers' and the external examiners' expectations of the structure, generality, and validity of proof influence their notions of what constitutes a proof. Students also consider the examination opportunities that require the development of reasoned explanations as possible opportunities to construct proof arguments.

The combined findings of these three studies could help researchers understand the implications of the recent reform recommendations on the teaching and learning of proof in Trinidad and Tobago. Firstly, these findings can be useful to policy makers and education stakeholders in their future efforts for developing the national curriculum, revision or development of instructional policies, and recommendations of textbooks and instructional support materials. Secondly, these findings can help curriculum designers, examiners, and teachers in creating future opportunities in the national curriculum and CSEC mathematics syllabus to support students' learning of proof in Trinidad and Tobago. Thirdly, these findings can help educational stakeholders understand the type of support that is needed for teachers' future professional development and students' competency with reasoning and proof on CSEC examinations. This international study is a case of the larger issues surrounding reform implications in a centralized governed educational system, which offers uniform prescriptive guidance for teaching and uniform curriculum support for learning. Furthermore this work potentially adds to the ongoing discussions in mathematics education about the interplay between policy, practice, and student learning.

ACKNOWLEDGEMENTS

Several individuals have supported me through the inception and completion of this dissertation. First I want to thank Associate Professor Gloriana González, my advisor and chair of my dissertation committee. Professor González has been an excellent mentor, counselor, and support throughout my doctoral studies. From the early discussions about my interest in mathematics education in Trinidad and Tobago, Professor González guided me, asking thought-provoking questions, critically analyzing my scholarship, providing insightful and thorough feedback, and motivating me to excel in my scholarship. I sincerely thank you for your ongoing enthusiasm to make me always autograph my work with excellence.

I also wish to thank my dissertation committee members: Professor Cameron McCarthy, Professor Tami Martin, and Associate Professor Monica McDermott. Each one of you have shared your expertise and provided insightful comments and meaningful feedback through the process of writing this dissertation. I value your time and efforts to mold me into the scholar who can make a vital contribution to research in mathematics education.

My dissertation study was successful due to the participation and cooperation of the teachers, Ms. V, Mr. P, and Ms. M. I thank you for allowing me the opportunity to enter your classroom and observe your teaching and interview your students. I value the excellent demonstrations of reform-based instruction evident in classrooms in Trinidad and Tobago. I am grateful for the opportunity to give you a forum to share with policy makers, educational stakeholders, and researchers about the intricacies of your daily routine and the dynamics between the curriculum, teaching, and your students' learning. I also thank the three schools that facilitated my data collection and the students, who eagerly participated in the student interviews.

I also wish to thank my research team, Mr. Dale Jones, research assistant, Mrs. Candy Springer-Blackman, transcriber, and Ms. Martha Makowski, reliability coder and colleague. I appreciate your time and expertise in assisting me in completing various phases of this dissertation.

My dissertation research was supported by a grant from the College of Education Hardie Dissertation Award Program. I appreciate the resources and support that afforded my data collection in Trinidad and Tobago and completion of my doctoral work. I also wish to thank the Fulbright Faculty Development Scholarship Program and the Latin American Scholarship Program for American Universities (LASPAU) for providing me the opportunity to pursue my masters and doctorate at the University of Illinois at Urbana-Champaign.

My family and friends have supported me immensely throughout this journey. My mother, Monica Cecily Hunte, continually provided encouragement and motivation to complete my work. I acknowledge your zeal for ensuring that I realized my deceased father, Arthur Hunte's aspiration for me to obtain my PhD. My sister Nicole, I thank you for your continuous calls and prayers. I am also grateful for the family that I have gained in the Benjamins. They have continually encouraged me and provided the necessary spiritual and emotional support to complete this venture. My friends and relatives in the US and Trinidad and Tobago, I deeply value your continued support and efforts in helping me complete my program and always believing in my potential to always excel in my endeavors. My church family, I thank you for your spiritual encouragement and also making Urbana-Champaign my home during my studies.

My wife, Patrice, as my life partner, number one cheerleader, and eternal love. Words cannot describe the ultimate sacrifice you made to accompany me to the US complete this milestone in my life. I acknowledge your continued support, coaching, care, and undying

encouragement during the most difficult stages of this work, I am eternally grateful. I also value your interest in my work as a fellow mathematics educator and I feel blessed to have such a wonderful spouse, friend, and supporter by my side.

TABLE OF CONTENTS

CHAPTER 1: INTRODUCTION…….1

CHAPTER 2: THE NATURE OF REASONING AND PROOF OPPORTUNTIES IN SECONDARY SCHOOL TEXTBOOKS IN TRINIDAD AND TOBAGO……...27

CHAPTER 3:THE TEACHING OF REASONING AND PROOF IN TRINIDAD AND TOBAGO……128

CHAPTER 4: SECONDARY SCHOOL STUDENTS' CONCEPTIONS OF PROOF IN TRINIDAD AND TOBAGO…….290

CHAPTER 5: CONCLUSION…….401

REFERENCES……..428

APPENDIX A: CLASSROOM OBSERVATION PROTOCOL …….449

APPENDIX B: PRE-OBSERVATION INTERVIEW PROTOCOL……...451

APPENDIX C: POST-OBSERVATION INTERVIEW PROTOCOL……452

APPENDIX D: PROOF OF WHY ASS OR SSA DOES NOT PROVE CONGRUENCE……453

APPENDIX E: SIX PRINCIPLES OF PROOF UNDERSTANDING ……454

APPENDIX F: STUDENT INTERVIEW PROTOCOL……...456

CHAPTER 1:

INTRODUCTION

In recent years mathematics educational reform efforts around the world have established ambitious goals for schools, teachers, and students. These reform efforts suggest changes to the curriculum content, instructional policies, assessments, students' opportunities to learn, and teachers' professional development (Borko & Putnam, 1998; Floden, 2002; Hiebert & Stigler, 2004). In Trinidad and Tobago, similar reform efforts in mathematics education suggest changes to all the aforementioned facets but the most significant change is the recommended change in instruction. By instruction, I consider the definition of Cohen, Raudenbush, and Ball (2003), which states that instruction consists of the three-way classroom interactions between teachers and students around content directed towards facilitating students' achievement of specified learning goals. According to these researchers, instruction is continuous and "it flows in and draws on environments-including other teachers and students, school leaders, parents, professionals, local districts, state agencies, and test and text publishers" (p. 122). In my dissertation, I use this definition of instruction to examine the implications of reform on teaching and student learning in Trinidad and Tobago. My overarching question is: What are the instructional implications of the reform in mathematics education in Trinidad and Tobago? To answer this question, I will use reasoning and proof as a case to examine these implications as they relate to the written curriculum materials, students' perceptions of the opportunities to do proofs in the written curriculum, and the sociomathematical norms surrounding teacher's instruction of reasoning and proof.

According to the reform documents in Trinidad and Tobago, teachers are expected to foster opportunities for students to develop deep understanding of important mathematics

content, to think critically, to construct and solve problems, to synthesize information, invent, create, express themselves proficiently, and leave school prepared to be responsible citizens. Classrooms are to have the type of setting in which teachers and all students engage in rich discourse about important ideas and explore problems framed in meaningful contexts. Students are expected to be active participants who construct knowledge under the guidance of their teacher (Republic of Trinidad and Tobago, Ministry of Education, 2003, 2009). This new vision of classrooms represents a substantial departure from the traditional classrooms in Trinidad and Tobago (Bell-Hutchinson 2004). The new vision of the reform is based on fundamentally different assumptions about content, teaching, and learning.

In this dissertation, I examine mathematics education reform in Trinidad and Tobago. I particularly inquire about the implications for the content in the curriculum, teachers' instructional practices, and students' learning, I use reasoning and proof as a case of examining these reform efforts. As a result, I focus on the notion of students' opportunities to learn reasoning and proof in curricular materials, student's perspectives of the new recommendations for reasoning and proof, and finally looking closely at teachers' creation and implementation of opportunities to learn reasoning and proof. In the next section, I discuss the idea of opportunities to learn reasoning and proof followed by my overarching framework for the dissertation study.

Opportunities to Learn Reasoning and Proof

In considering the implications of the reform movement on student learning, an important construct I consider is "opportunities to learn." This construct refers to the conditions that allow students to do an academic task in order to achieve learning objectives (Hiebert & Grouws, 2007). Several researches used this concept to help explain the differences in achievement among students (e.g., Floden, 2002; Husen, 1967); and also to account for differences in

students' mathematics learning across different curriculum contexts between countries on international comparisons such as the Trends in International Mathematics and Science Study TIMMSS (Stigler & Hiebert, 1997). In the same manner, the implications of the reform on students' opportunities to learn reasoning and proof could inevitably impact their performance on terminal examinations assessment of these objectives.

The policy documents in Trinidad and Tobago, recommend that that students should be given opportunities to develop critical thinking and reasoning skills and that teaching should involve guiding students in the process of making sense of mathematical ideas (Trinidad and Tobago, 2005, 2006). This recommendation suggests that students' opportunities to learn reasoning and proof will directly impact their achievement of these learning objectives. Students' opportunities to learn reasoning and proof will be influenced by several factors. These factors include the resources for teaching proof that can affect teachers' decisions, the emphasis teachers place on the reasoning and proof, the amount of time allocated for proof-related and problem solving activities, and teachers' perspectives of student learning whether constructivist or behaviorist. The latter factor will influence the type of tasks teachers give their students, their decision-making during instruction, the type of questions they pose, the responses they expect from students, and the nature of the discourse in their classroom. Furthermore the curriculum teachers use, influences students' opportunity to learn (Hiebert & Grouws, 2007; Stein, Remillard, & Smith, 2007). Therefore the new reform-oriented curriculum and textbooks must provide sufficient support for teachers to create the conditions deemed necessary for reasoning and proof to be an integral part of secondary school students mathematical experiences. In the case of mathematics education in Trinidad and Tobago, we are yet to know how the reform-

oriented mathematics curricula and teachers, as agents of the reform, help enhance students' opportunities to learn reasoning and proof in secondary school.

Overarching Framework

In this dissertation study, I use Cohen, Raudenbush, and Ball's (2003) description of the instructional triangle to guide my examination of the implications of the reform efforts on instructional practices and student's learning in Trinidad and Tobago. My overall investigation looks at the opportunities to learn reasoning and proof as it relates to the three components of the instructional triangle. The three components include:

1. Content: I analyze the recommended textbooks to determine the type of support available for the teaching and learning of reasoning and proof.
2. Teaching: I examine the nature of the teaching of reasoning and proof in classrooms. I am interested in teachers' instructional practices, particularly, the sociomathematical norms surrounding their creation and management of opportunities to learn reasoning and proof.
3. Students' Learning: I look for evidence of students learning about proof. I gather the evidence by analyzing students' perspectives about what constitutes a proof , the roles of proof and opportunities to learn in the curriculum materials.

Figure 1.1 shows the elements of focus for examining reform in this dissertation study. The intersecting circles represent the three components, (i.e., content, teachers, and students) of Cohen, Raudenbush and Ball's (2003) definition of instruction. In particular, I examine evidence of students' learning and teachers' instructional practices with regard to the student and teacher components of the instructional triangle. In Figure 1.1, the intersection of the three components contains reasoning and proof, which is the case of my examination of the implications of reform

in Trinidad and Tobago. Figure 1.1 also shows how my study will be divided into three smaller studies. Each of these studies in the dissertation corresponds to the interaction between the components of the framework. In the following section I describe the components of mathematics education in Trinidad and Tobago and how it presents a unique case for my dissertation study.

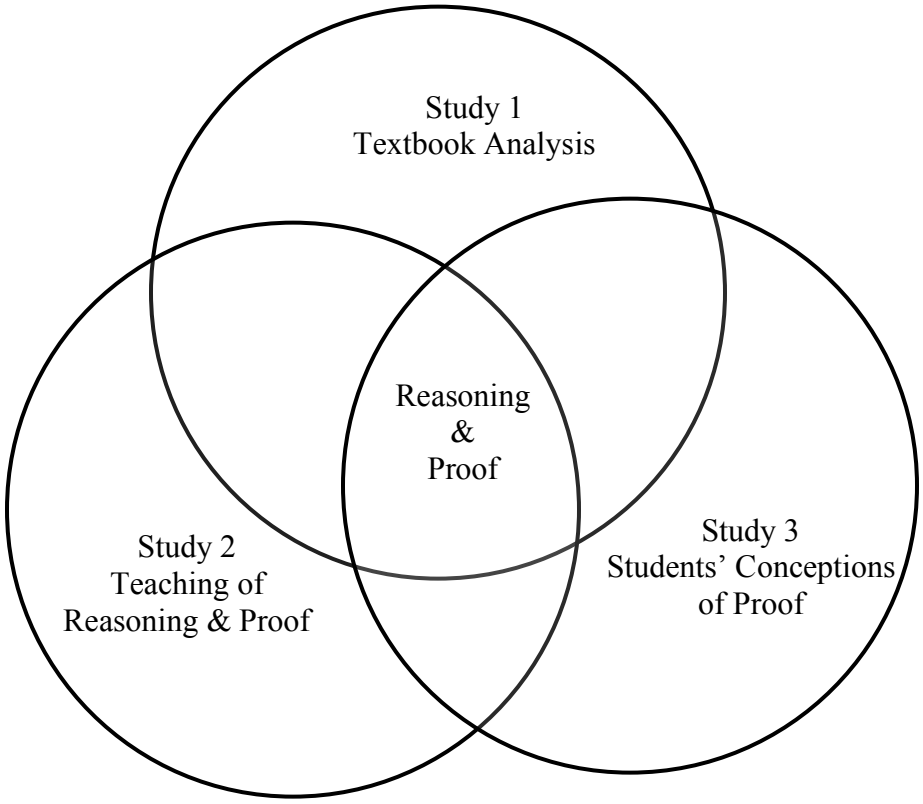


Figure 1.1: Elements of focus for examining mathematics reform in Trinidad and Tobago.

Education in Trinidad and Tobago

Trinidad and Tobago is a twin island Republic at the southern most end of the Caribbean islands. It has a multicultural population, a consequence of a history of various agricultural labor endeavors, namely slavery and various forms of indentureship during successive periods of

Spanish, French, and British colonization. The strong cultural influence of these colonial ties is evident in the language, ethnicity, and education system of Trinidad and Tobago.

In Trinidad and Tobago, there were two major educational reform movements. The first occurred in the 1970's as an attempt to break colonial ties by the adoption of internal examinations in the Caribbean, through the formation of the Caribbean Examination Council (CXC) and the establishment of national training colleges for elementary and secondary level teachers (London, 1997). The second movement in the 2000's, which will be the focus of my dissertation, was funded by the Inter-American Bank to aid in the government's program for modernization of the secondary school system. This reform effort was part of the government's strategic plan to elevate the status of Trinidad and Tobago as a leader in education and innovation in the developing world (Trinidad and Tobago, 2005; Wolff & Castro, 2000).

The main focus of the government's education policy was to create a modern system that can provide the education, training, and values relevant to the developmental needs of the country. In this regard, the government placed high priority on the improvement of the quality of primary and secondary education. These initiatives in education were expected to create an education system whose contents and methods reflect the social and cultural realities and provide young people with skills for living, working, and citizenship (Trinidad and Tobago, 2005, 2007). Several researchers claim that educational reform policies primarily seek to increase student achievement by manipulating elements of instruction, curriculum, assessments, teachers' knowledge, and practices (e.g., Cohen & Hill, 2000; Hiebert & Stigler, 2004). Educational researchers in Trinidad and Tobago are yet to understand how the implications of the new vision of a student-centered approach to learning and teaching coupled with the suggested changes in assessment and content, will have on the teaching practices and student learning. My dissertation

study will help fill this gap as it relates specifically to Trinidad and Tobago and can help mathematics educators and policy makers understand the dynamics of the relationship between policy and practice.

Administration and Management of Education

The Ministry of Education of Trinidad and Tobago (MOE) administers and manages all educational policies of elementary, secondary, and post-secondary education in Trinidad and Tobago. Centralization is the traditional mode of decision‑making in the education system in Trinidad and Tobago, (London, 1997*)*. The Curriculum Planning and Development Division of MOE handles the administrative processes related to curriculum development, implementation, and reform. In particular, the Secondary Education Modernization Program (SEMP) determined the objectives of the recent curriculum reform efforts. The MOE, in conjunction with CXC, prescribes curricula, textbooks, and stipulates the requirements that determine the successful completion of secondary school. Stakeholders and the government provide the funding for the purchasing of textbooks, workshops for teachers, and materials for each suggested curriculum guide for core subjects[1]. Teachers are accountable to the MOE, whereas students are accountable to their teachers, the National Examination Council (NEC), and CXC, who are responsible for the various forms of formative and summative assessments.

The Education System of Trinidad and Tobago

The education system in Trinidad and Tobago comprises five levels, namely pre-primary (ages 3 to 5), primary (ages 5 to 11), secondary (ages 11 to 16), post-secondary (ages 16 to 18), and tertiary level. Both government schools and assisted (denominational) schools provide

[1] The core subjects are Language Arts, Mathematics, Science, Social Studies, Visual and Performing Arts, Spanish, Technology Education, Physical Education, Agricultural Science, and Industrial Arts or Home Economics.

primary and secondary education. The terms denominational and assisted are synonymous titles for the schools in this category. The key difference between government and assisted schools is that government schools are fully funded and operated by the government while private bodies, usually religious boards, manage assisted schools and at least half of their expenses are paid for by the government (Jackson, 2012). While assisted schools are often considered more elite, along all other facets government and assisted schools are the same.

At the end of primary school (Standard 5), students take the Secondary Entrance Assessment (SEA) and are assigned to secondary schools based on scores on this exam and a list of four ranked school choices by the Ministry of Education. Secondary school begins in Form 1 (ages 11 to 12) and ends at Form 5 (ages 15 to 16) when students take the Caribbean Secondary Education Certification (CSEC) examinations.[2] Members of CXC externally grade these exams. Students seeking to continue their education typically take five or more subjects, and all students take the English language and Mathematics exam. The CSEC examinations are accepted as an entry qualification for higher education in Canada, the United Kingdom, and the United States.

A successful completion of CSEC examinations gives entry into post-secondary institutions. Students can choose to pursue two-year advanced proficiency courses in Form 6 (ages 16 to 18) in core subject areas associated with their future majors at college. Advanced level students take the Caribbean Advanced Proficiency Examination (CAPE) in their selected subjects. After completion of secondary school, students also have the option to move on to a variety of both public and private training or vocational institutions offering a range of programs leading to diplomas and certificates. Advanced level graduates can enroll directly at the national

[2] There are 31 CSEC subjects covering a range of purely academic subjects such as Physics, Chemistry and Geography, and more work and vocationally related subjects such as Technical Drawing and Principles of Business and Office Procedures.

or regional institutions (e.g., University of the West Indies, University of the Southern Caribbean, and University of Trinidad and Tobago) or study abroad.

The Mathematics Curriculum

The Secondary Mathematics curriculum in Trinidad and Tobago has three levels: Lower Secondary in Forms 1- 3 (ages 11 to 14), Upper Secondary in Forms 4 and 5 (ages 14 to 16), and Post-Secondary in Form 6 (ages 16 to 18). CXC administers the two major terminal assessments of students' mathematical knowledge at the Upper Secondary and the Post-Secondary levels. At the Upper Secondary level, all students take the mandatory CSEC examination and students at the Post-Secondary level take the Caribbean Advanced Proficiency Exam (CAPE). All students write the National Examination Council (NEC) exam in the final year of their Lower-Secondary School (Form 3). Although schools have the option of selecting suitable resource materials for each level, their options are usually restricted by the offerings in the MOE approved textbook listing and rental system. The MOE consults with external examining bodies CXC and NEC to ascertain suitable textbooks for their listing. As a result there exists uniformity in the textbooks adopted throughout the country for each form level in secondary school.

Trinidad and Tobago as a Case for Examining Reform

Trinidad and Tobago has a centralized education system with a national curriculum and assessment. In fact, Trinidad and Tobago provides a good case to explore the phenomenon of reform and its effects because of its centralized governed educational system, centralized mathematics content, and collective set of instructional and assessment resources such as standardized textbooks and examinations. These characteristics provide a common ground for investigating educational issues, and working with stakeholders to solve problems of teaching and learning.

Researchers examining reform efforts in other countries indicate that the fragmented design of school governance and the lack of homogeneity in the knowledge and assessments create issues for teachers meeting the demands of the reform movement (Cohen, 2011; Cohen & Hill, 2000; Lubienski, 2011). For example Cohen (2011) suggests that the lack of a national curriculum or centralized governing body in the United States has possibly led to a plethora of issues for teachers in the classroom. These issues include (a) multiple interpretations of what counts as valuable knowledge for students' success within and between educational districts, (b) the varying standards for acceptable teacher and student performance, and (c) the political disintegration of school governance that fails to hold teachers and students accountable at an uniformed national level. Cohen views these as characteristics of the U.S. education system that hinder the successful implementation of reform policies. He hypothesizes that a centralized governing body with a national curriculum and assessments may provide a common framework for teachers to develop their practice and help educational stakeholders make valid judgments about students' progress and performance. Lubienski (2011) noted that the most challenging tasks of mathematics education reformers is to design initiatives that support teachers' development of equitable classroom practices that align with the reformer's vision for teaching and learning. Curriculum materials such as textbooks are the means through which some policy makers have provided support to teachers for meeting the objectives for reform-based instruction (Borko & Putnam, 1998). In agreement, Lubienski (2011) and Cohen (2011) also identified textbooks as a major vehicle for helping teachers meet the instructional demands of the reform.

Because of the influence of textbooks in teachers' response to the reformers' recommendations, it is crucial to consider the role of textbooks and curriculum resources in the reform movement in a centralized educational system. Textbooks can be key instruments in

Trinidad and Tobago's reform movement to align mathematics instruction with reform recommendations. Textbooks could also vary in their effectiveness as agents of change. In Trinidad and Tobago, the school system offers very prescriptive guidance for content coverage (Jennings 1993; Lewin, Keller, & Taylor, 2002). In addition to required courses, centralized governing bodies such as MOE prescribe instructional resources. In this case, textbooks and curriculum guides offer extensive and focused guidance about instructional content. Several researchers claim that in such education situations it appears that teachers attend to the guidance, and that content coverage is relatively homogeneous (Lubienski, 2011; Otten, Gilbertson, Males, & Clark, 2014; Porter, Floden, Freeman, Schmidt, & Schwille, 1988; Schwille et al., 1983). The uniformity of the prescription of textbooks and teaching materials may lead to textbooks being a effective agent of reform policy in Trinidad and Tobago. Therefore, Trinidad and Tobago provides a good case to examine the role of curriculum materials in helping teachers meet the demands of the reform. The centralized governance, national policies, curriculum materials, and assessments provide an interesting context to look at the dynamics of the reform on the teaching and learning of mathematics.

Mathematics Education Reform in Trinidad and Tobago

Recent curriculum reform efforts in Trinidad and Tobago have addressed educators' increased focus on the limited mathematical competence of secondary school students. Reports on mathematical achievement from external examination bodies, the National Task Force on Education Report and MOE have independently concluded that employers in public and private agencies believe that the majority of students at both primary and secondary levels lack basic skills in deductive reasoning (Republic of Trinidad and Tobago, Ministry of Education, 2009). The National Task Force Report (1994) claimed that,

> The failure in our school mathematics is due to traditions of teaching that are inappropriate to the way most of our students learn that is through construction of their own understanding based on their everyday experiences in and out of the classroom. (p. 12)

In 1996, the government of Trinidad and Tobago adopted the National Task Force report as the national policy on education and used its recommendations to improve secondary education. As a result of this adoption, the government created the Secondary Education Modernization Programme (SEMP), to modernize secondary education in Trinidad and Tobago. The main objective of this program was to reform the national curriculum in the core subjects and facilitate the adoption of assessments through CXC and recommend textbooks that support the objectives in the reform-oriented national curriculum.

The rationale for reform in mathematics education. According to the authors of the recent reform document, mathematics is seen as an activity that is intrinsic to the development of the mind, civilization, and the daily lives of individuals. It is viewed as a powerful form of communication that enables learners to represent, interpret, explain phenomena, and make sound predictions (Republic of Trinidad and Tobago, Ministry of Education, 2003, 2009). The reformers claim that the mathematics curriculum should offer opportunities for students to become logical, creative, and critical thinkers through the development of logical thinking, problem solving, strategic thinking, and investigative skills. The ultimate mission outlined by the reform documents is that the new mathematics curriculum should foster "the growth and development of mathematically empowered students in Trinidad and Tobago so that they can effectively contribute to our society and serve as catalysts to world development" (Republic of Trinidad and Tobago, Ministry of Education 2009, § 2: 1). Mathematics is viewed as an agent of

developing innovative citizens that can actively contribute to the country's development.

Problem solving and proving in reform efforts. One of the six learning objectives outlined in the recent reform-oriented national curriculum is titled Problem Solving. This learning objective is considered as the only area that offers some support for the inclusion of reasoning and proof in all secondary school students' mathematical experiences. According to the policy documents "mathematics classroom activities must be structured to provide opportunities for students to communicate, reason mathematically, make connections, make conjectures, share solutions, and verify reasoning" (Republic of Trinidad and Tobago, Ministry of Education, 2003, § 3:3). This quote highlights the underlying goal of the reform, which is the notion of enhancing students' understanding of mathematics and helping them explore patterns, solve problems, and justify their processes of reasoning. Indeed the goal of problem solving may not directly imply increased proof and reasoning opportunities, however there exists an inextricable link between proof and problem solving.

According to several researchers, the mathematical processes of proving and problem solving are closely linked (e.g. Balacheff, 2010; Lucast, 2003; Schoenfeld, 1992). For example, Balacheff (2010) noted that a proof could be considered as a visual representation of one's reasoning during problem solving. A written proof is a way of expressing the justifications or explanations of one's ideas when solving problems. Lucast (2003) also held a similar view that agrees with Balacheff in that he stated, "the proof written in the end is the clean version of all the thinking that has taken place" (p. 101). This quote highlights earlier assertions made by Schoenfeld (1992), who claimed that during problem solving, students learn content through the acquisition of mathematical tools used for working with and applying mathematical knowledge. Schoenfeld claimed that this acquisition helps students make sense of mathematical ideas and

allow them to make arguments, which express their thinking, in appropriate and flexible ways. As a result, students could be expected to logically organize their thinking into a more formal format of a mathematical proof at the end of their problems solving activity. The aforementioned arguments support an existing relationship between problem solving, proof, and reasoning in mathematics.

In Trinidad and Tobago's reform-oriented curricular documents, the authors describe problem solving as a dynamic process of pattern identification, conjecturing, and proving. The authors' orientation is similar to what was proposed by Pólya's (1957) arguments, which claimed that the aforementioned actions establish the inextricable relationship between proving and problem solving when a student extends their solutions of particular cases to the proof of more general cases. Based on the close relationship between proof and problem solving one could assert that the reform movements inclusion of a problem solving learning outcome in Trinidad and Tobago, reflects a focus on making proof an integral part of students' mathematical learning during problem solving activities.

Teaching strategies and assessment. The reform movement in Trinidad and Tobago could be described by the major thrust in creating the new national mathematics curriculum. This consisted of the creation of teaching guides which present the reformers' philosophy for instructional policy, assessment, and to some extent professional development. Although the Curriculum, Planning and Development unit of the MOE led the writing process of these guides, a team of mathematics educators from the major universities and teacher training institutions along with a selected group of 21 teachers, designed and revised the curriculum guides. It is the intention of the reformers that along with the CSEC mathematics syllabus teachers receive

guidance in meeting the reformers demands and helping their students successfully matriculate through the CSEC mathematics course[3].

The content of the new national mathematics curriculum includes both core content and extensions. The extensions consist of optional mathematical concepts, which facilitate students' further understanding of material in the core mathematics syllabus. For example, in the extensions unit in Geometry for Form 4 and 5 (ages 14 to 16), students learn about three-dimensional visualization of shapes to help support their reasoning in formulating algebraic expressions for the volume and surface areas of various objects. This type of reasoning helps them solve real-world problems such as the water capacity of a water tank for a household. The activities in this extension are intended to help students make inferences and generalizations from given data. Every student in Trinidad and Tobago is expected to master at least the material in the core curriculum. A student, who has mastered the material in both the core and extensions, should be able to perform well in mathematics at the CSEC examination and other comparable examinations offered by other examining bodies. Students, who master the recommended skills and suggested core content, should be well positioned to attain the overall outcomes targeted in the curriculum guides (Republic of Trinidad and Tobago, Ministry of Education, 2009). Overall the reform-oriented national curriculum presents the statements of aims and the expectations of teachers in achieving the above mentioned six learning objectives. This document also presents a suggested scheduled program of how the mathematics content areas should be covered for each grade level in secondary school.

The CSEC syllabus also provides guidance to teachers and is complementary to the teaching guides in the new curriculum. The CSEC mathematics syllabus includes examination

[3] In Trinidad and Tobago the mathematics syllabus outlines the requirements that students must complete in order to successful pass the CSEC examinations.

guidelines, sample questions and solutions along with School-Based Assessments (SBA)[4]. The CSEC mathematics syllabus contains the learning objectives for each content area that CXC assesses at the end of the fifth year of secondary school in the CSEC mathematics examination. CXC also provide teachers with guidelines about how each past examination questions were graded and suggestions for future preparation of students in the respective subject's annual reports. The curriculum and the CXC syllabus are intended to take learners through a spiraling learning process that integrates learning through content and pedagogy. These two documents can be used primarily for developing the necessary competencies and understandings that will help students to attain the six essential learning outcomes of the reform movement.

Reform Efforts Concerning Reasoning and Proof in Trinidad and Tobago

Recent reform efforts in Trinidad and Tobago recommend that reasoning and proof become an integral part of secondary school students' mathematical experiences. By reasoning and proof I refer to two separate notions. Reasoning is the practice of looking for relationships, making conjectures, providing deductive arguments, and providing other evidence to support a mathematical statement (Hanna & Jahnke, 1996). Whereas, proof is the mathematical result of using accepted rules of inference to construct arguments that establish the validity of mathematical statements (Reid, 2001, 2005). Reasoning and proof in mathematics play an instrumental role in helping students experience the work of mathematicians (Pólya, 1957). In mathematics classrooms, reasoning and proof activities provide opportunities for students to develop an understanding of how mathematicians make assertions and to explain why these mathematical assertions are true. The construction of proofs and mathematical arguments require the ability to reason mathematically, which is to make inferences and deductions from

[4] SBA projects are a type of continuous assessment scheme used by CXC at all levels of examinations at secondary school for validating students' competency and progress.

mathematical structures (Küchemann & Hoyles, 2009). Students' work with proof in mathematics classroom leads to the development of logical reasoning and critical thinking skills and understanding the importance of proving activities to the discipline of mathematics.

According to the curricular documents "secondary school mathematics teachers must provide students with opportunities that reflect the nature and role of proof in mathematics " (Republic of Trinidad and Tobago Ministry of Education, 2003, § 3:3). The reform recommendations suggest that secondary school mathematics teachers have an influential role in enhancing the role of proof in their classrooms. Teachers are expected to offer opportunities for their students to understand the function, meaning, and purpose of proof in their mathematical experiences.

As teachers attempt to meet the reformer's vision for teaching and learning they encounter different demands. These demands are related to the (a) schooling, (b) reform expectations concerning the teaching and learning of proof, and (c) the discipline of mathematics. These demands are different in that the demands of schooling are framed within the ideas of what mathematical content, skills, and requirements students should possess for completion of secondary school and their success in terminal examinations. The demands of reform consist of recommendations and ideologies of policy makers and curriculum designers that are based on the knowledge and skills they consider each successful student should obtain from the secondary school mathematics curriculum. Furthermore, within the demands of the discipline of mathematics, opportunities for proving should fundamentally allow students to experience how conjectures are made and verified, and how mathematics is explained, explored, discovered, systemized, and communicated.

The development of reasoning and proof competency is a major objective of the reform policies in Trinidad and Tobago. This competency includes the skill to reason systematically and coherently, to identify valid and invalid arguments, to establish and construct valid mathematical proofs (Republic of Trinidad and Tobago, Ministry of Education, 2009). Reasoning and proof within a school context involves two kinds of reasoning: plausible and deductive (Pólya, 1954; Stylianides, 2009). Plausible reasoning is the type of mathematical activity whereby conjectures are generated. Plausible reasoning involves what Pólya (1954) calls "mathematics in the making" and frequently consists of the identification and arrangement of facts into mathematical patterns; the use of these patterns to make conjectures, and the testing of these conjecture against new empirical evidence. Deductive reasoning is the use of mathematical arguments to validate mathematical knowledge. Pólya (1954) noted that deductive reasoning is the result of a mathematician's creative work. A mathematical proof is a result of deductive reasoning but plausible reasoning discovers the proof.

Reasoning and proof as a case of examining reform. Proof is an integral component of mathematics (Hanna, 1995, 2000; Hersh, 2009). Several mathematics educators have claimed that students' opportunities with reasoning and proof are crucial to their development of deeper conceptual understanding of mathematics (e.g., de Villiers, 1999; Dreyfus, 1999). In recent years, there is a growing emphasis on proof in school mathematics, primarily because proof is the basis of mathematical understanding and is essential for developing, establishing, and communicating mathematical knowledge (Dickerson, 2008; Fujita & Jones, 2014; Healy & Hoyles, 2000; Knuth, 2002; Stylianides, 2009). This increased focus on proof is reflected in many countries as researchers and policy makers make several calls to make proof an integral component of all students experiences in school mathematics (e.g., Ball & Bass, 2003; Ball, Hoyles, Jahnke, &

Movshovitz-Hadar, 2002; Marrades & Gutiérrez, 2000; NCTM, 2000; Schoenfeld, 1994; Sowder & Harel, 1998; Yackel & Hanna, 2003). Given the recent reform recommendations to enhance the nature and role of proof in students mathematical experiences and the role of teachers in meeting the demands of the discipline, schooling, and reform; proof as a goal in mathematics education provides an opportunity for me to explore how teachers make their decisions during instruction as they are faced with demands of reform efforts, demands of the discipline of mathematics and demands of schooling. Reasoning and proof is fundamental to school mathematics (Hanna, 2000; Knuth, 2002), thus it provides a case of examining the bigger phenomenon of issues of reform and it effects on the teaching and learning of secondary school mathematics.

The Implications of Mathematics Education Reform in Trinidad and Tobago

Over the past two decades several studies have discussed education reform in the Caribbean, which are relevant to the study of reform and its effects in Trinidad and Tobago (e.g., George, Mohammed, & Quamina-Aiyejina, 2003; Jennings 1993; London, 1997). For example, one study demonstrated that teachers' identity, content knowledge, and teaching experience, influenced how they used, interpreted, and implemented reform-oriented instructional policies (George, Mohammed, & Quamina-Aiyejina, 2003). Another study explored how the resulting reform-oriented curricular responded only to the instructional philosophies of the central governing bodies such as the MOE and CXC with little or no input from teachers (Jennings, 1993). This study also claimed that teachers believed that their involvement in the curriculum reform was constrained by a host of politically based administrative protocols thus affecting their ability to endorse and enact reform recommendations. Additionally, Jennings's study highlighted the low pass rates in terminal exams, and suggested the need for an evaluative system that

includes observations of how teachers use and interpret terminal examination syllabi, as part of the system of accountability for student performance.

The aforementioned examples of research show the potential issues surrounding reform and its effects on teachers' use and interpretation of reform-oriented curricula. These examples also suggest the need for studies that examine the dynamics of reform efforts in secondary schools in Trinidad and Tobago, and, in particular, the examination of how teachers respond to the instructional policies of the reform. In this dissertation, reform provides the context by which I examine the possible implications on teaching and learning of mathematics. Although it is possible that the reform policies directly influence the teaching evident in classrooms, I posit that the teaching I observe may not be a consequence of the reform policies. With this hypothesis I do not take a stance of cause and effect, but I describe the current situation of teaching and learning framed within the recent reform efforts in Trinidad and Tobago.

Research Questions

To examine the implications of mathematics reform on the teaching and learning of reasoning and proof, I focus on three main areas, (a) opportunities for reasoning and proof in the textbooks, (b) the teaching of reasoning and proof, and (c) students' conceptions of reasoning and proof. Each of the aforementioned focus areas corresponds to one of three sub-studies that constitute this dissertation. In Table 1.1, I present the overarching goal for each study. I also list the research questions that will guide the data collection and analysis procedures for achieving the goal of each study. The main goal and research questions for this dissertation are:

Study 1: *The Nature of Reasoning and Proof Opportunities in Secondary School Textbooks in Trinidad and Tobago.*

Goal: To determine the nature and extent of the proof-related opportunities in the Geometry content of the textbooks in Trinidad and Tobago.

Research Questions:

1.1 What is the nature and distribution of the opportunities for reasoning and proof within the Geometry sections of secondary school textbooks in Trinidad and Tobago?

1.2 How do the reasoning and proof opportunities in Geometry sections of the textbook expositions compare to the opportunities in student exercises?

Study 2: *The Teaching of Reasoning and Proof in Trinidad and Tobago.*

Goal: To describe the instructional and classroom practices during the creation of opportunities to learn reasoning and proof in Geometry in Trinidad and Tobago.

Research Questions:

2.1 How do teachers implement proof-based Geometry mathematical tasks in their classrooms?

2.2 What sociomathematical norms do teachers establish during the teaching of reasoning and proof?

2.3 How do teachers use curricular materials (e.g., textbooks and exam preparation materials) to support the teaching of reasoning and proof?

Study 3: *Secondary School Students' Conceptions of Proof in Trinidad and Tobago.*

Goal: To investigate the nature of Secondary School students' conceptions of proof and in Trinidad and Tobago?

Sub-questions:

3.1 How do students in Trinidad and Tobago view the purposes of proof in secondary school mathematics?

3.2 How do students view the generality and validity of a proof?

3.3 What are students' perspectives about the opportunities for proving in Geometry in the CSEC examinations?

Table 1.1

Outline of three-paper model of dissertation study

Dissertation Study	Research Questions	Data Sources	Methods	Theoretical Framework
1. The Nature of Reasoning and Proof Opportunities in Secondary School Textbooks in Trinidad and Tobago.	1.1.What is the nature and distribution of the opportunities for reasoning and proof within the Geometry sections of secondary school textbooks in Trinidad and Tobago? 1.2. How do the reasoning and proof opportunities in Geometry sections of the textbook expositions compare to the opportunities in student exercises?	The three most recommended mathematics textbooks: Greer, A., & Layne, C. (1994). *Certificate Mathematics*. England: Stanley Thornes. Toolsie, R. (2009) *Mathematics a complete course.* Trinidad, West Indies: Caribbean Educational. Chandler, S., Smith, E., Ali, F. W., Layne, C., & Mothersill, A. (2008) *Mathematics for CSEC*. United Kingdom: Nelson Thornes.	Textbook analysis	Otten, Gilbertson, Males, and Clark (2014)

2. The Teaching of Reasoning and Proof in Trinidad and Tobago.	2.1. How do teachers implement proof-based Geometry mathematical tasks in their classrooms? 2.2. What sociomathematical norms do teachers establish during the teaching of reasoning and proof? 2.3. How do teachers use curricular materials (e.g., textbooks and exam preparation materials) to support the teaching of reasoning and proof?	Classroom observations and pre-lesson and post-lesson interviews with 3 teachers. Teachers will teach lessons covering *Congruency of Triangles*.	Case studies	Sociomathematical norms (Yackel & Cobb, 1996), Characteristics of reform-teaching Hufferd-Ackles, Fuson, & Sherin, 2004
3. Secondary School Students' Conceptions of Proof in Trinidad and Tobago.	3.1. How do students in Trinidad and Tobago view the purposes of proof in secondary school mathematics? 3.2. How do students view the generality and validity of a proof? 3.3. What are students' perspectives about the type opportunities for proving in the CSEC examinations?	Student interviews with 21 students.	Semi-structured Interviews	Six Principles of understanding proof (McCrone & Martin, 2009)

Outline of Dissertation Study

My examination consist of three studies which focus on the opportunities for reasoning and proof in written curriculum materials, a description of three teachers during their teaching of proof-based lessons and students' perceptions of proof. In Table 1.1, I outline the three studies, the research questions, theoretical framework, and sources of data. In the first study I analyze the Geometry content in three textbooks. The second study entails a description of the teaching practices surrounding instruction of reasoning and proof. The final study of the dissertation entails an investigation into students' conceptions of proof. The three studies in my dissertation are layered in that the findings of each study will inform the design and rationales for the subsequent studies. For example, my analysis of the opportunities for reasoning and proof in the textbooks will guide my interview questions for students in the third study of my dissertation. This is useful so as identify any apparent consonances between my findings and existing literature on students' performance and conceptions with respect to reasoning and proof. I also use the findings of my analysis of textbooks to guide my examination of teachers' use of the textbooks during instruction in the second study. This is important to determine the existence of possible links between reasoning and proof opportunities in curriculum materials and the reasoning and proof activities offered during teacher's instruction.

Due to the layered nature of the studies, throughout this dissertation there will be repetition of themes and contextual descriptions of the reform in Trinidad and Tobago. In particular, I describe the aspects of the mathematics education reform that are pertinent to my inquiry in each study. Although each chapter builds on the findings of previous chapter, it is recommended that readers who plan to read the entire document should read the chapters in order. However, because the chapters 2, 3, and 4 were written as independent studies, they can

be read in any order. In the following chapters, I present each of the studies that constitute this dissertation. I include the research questions, literature review, theoretical framework, methodology, findings and discussion. In chapter 5, I conclude my discussions by presenting the overarching implications of the three studies. I also discuss the limitations of the study and directions for future research. At the end of the dissertation, I include a single list of references to avoid unnecessary repetition.

CHAPTER 2:

THE NATURE OF REASONING AND PROOF OPPORTUNITIES IN SECONDARY SCHOOL TEXTBOOKS IN TRINIDAD AND TOBAGO

Introduction

In Trinidad and Tobago, the reformers' recommend that reasoning and proof should be an important component of students' opportunities to learn mathematics. However, researchers are yet to understand how these mathematical practices are promoted in the recommended textbooks. To understand the nature of the opportunities for reasoning and proof, I employed an analytical framework created by Otten, Gilbertson, Males, and Clark (2014) to examine the nature and extent of opportunities for reasoning and proof in three textbooks. The findings of this examination can potentially help to design future textbooks and terminal examinations preparatory materials that offer opportunities for reasoning and proof.

Several researchers claim that textbooks are an important influence on students' educational experiences in school mathematics (e.g., Cohen, Raudenbush & Ball, 2003; Grouws, Smith, & Sztajn, 2004; Moyer, Cai, Wang, & Nie, 2011; Schmidt et al., 2001; Stein, Remillard, & Smith, 2007). Textbooks influence what students learn, when they learn it, and how well they learn it. On a global perspective, researchers report that efforts to change the content of the school curriculum, in particular textbooks has been viewed and used as an effective way to influence instructional practices, student learning, and meet the recommendations of curriculum reform (Cai & Cirillo, 2014; Cai & Howson, 2013; Senk & Thompson, 2003). Several studies, including the Third International Mathematics and Science Study (TIMSS), have shown that textbooks continue to play an important role in classrooms around the world (e.g., Fujita & Jones, 2014; Stylianides, 2009; Valverde et al., 2002). Therefore, textbooks have been called a

vehicle of change for educational reform (Ball & Cohen, 1996). Mathematics textbooks can play a vital role in students' opportunities to engage in reasoning and proof; and also convey the many decisions that teachers make about the construction and execution of mathematical opportunities offered to their students (Grouws, Smith, & Sztajn, 2004; Stylianides, 2007, 2009). Despite these efforts to make reasoning and proving central to school mathematics in Trinidad and Tobago, there are no existing studies that investigate how secondary school mathematics textbooks promote reasoning and proof.

Conceptualization of Reasoning and Proof

In this study I used the conceptualization of reasoning and proof developed by Stylianides (2009). Therefore, I refer to reasoning and proof as the mathematical activities of (a) pattern identification, (b) conjecturing, (c) providing non-proof arguments, and (d) constructing proofs. Following Stylianides (2009), I refer to pattern identification as the task of identifying a "general mathematical relation that fits a given set of data" (p. 263). For example, within this mathematical activity, students in Geometry can firstly examine several cases of geometrical objects. Secondly, students can create a data set and then find a general geometrical relation that aptly describes the data set. At the end, students identify a geometrical pattern as the first activity within reasoning and proof.

In the second activity of conjecturing, I refer to the mathematical activity of constructing and testing conjectures. Stylianides (2009) defined a conjecture as "a logical hypothesis about a general mathematical relation, which is based on incomplete evidence" (p. 264). The construction of conjectures refers to the actual development of hypotheses about a generalized mathematical relation with some measure of uncertainty about the validity of the hypothesis. The testing of conjectures entails empirical explorations, where a few examples are used to

investigate the validity of the hypotheses. In Geometry, students may observe a generalized pattern after exploring several geometrical objects. As a result, students may make a hypothesis (conjecture) describing the generalized observed pattern. At this stage students may begin to test whether their conjectures hold by testing several sets of geometrical objects.

In the third activity, the development of non-proof arguments pertains to the use of empirical examples and rationales to support one's judgments about the validity of a conjecture. Sentences, diagrams, and examples can be used to construct non-proof arguments. The non-proof arguments could also include and not limited to the use of non-mathematical language, which explains one's reasoning about how and why a conjecture or mathematical claim may be valid. Overall, a non-proof argument is an argument missing some logical deductions in its structure (Stylianides, 2009). A non-proof argument may lack some of the logical deductive arguments that connect the hypotheses to the conclusion.

The final activity is the construction of a proof. A mathematical proof is "a formal way of expressing one's reasoning and justification" (NCTM, 2000, p. 56). Proof as defined by Stylianides (2007) "is a valid argument based on accepted truths for or against a mathematical claim" (p. 195). By an argument, Stylianides referred to a connected sequence of claims. The validity of the argument is determined by accepted canons of mathematical inferences such as modus tollens and modus ponens. The accepted truths that govern the construction of the proof include axioms, theorems, definitions, and modes of reasoning shared by a community such as a group of mathematicians or a classroom of students. The construction of a proof is considered an individual activity framed by the shared understanding of the accepted truths, and criteria for validity defined by a mathematical community.

Proof is considered the final product of reasoning activities such as pattern identification and conjecturing (Hanna, 2005; Stylianides, 2009; Thompson, Senk, & Johnson, 2012). During reasoning activities, students make sense of patterns or conjectures, which eventually lead to developing non-proof arguments or proofs that support their sense making (Reid, 2001, 2005). By proving, I refer to the activities that lead to the construction of proofs. More specifically proving is the process of developing arguments consisting of logically rigorous deductions of conclusions from hypotheses (NCTM, 2000). Whereas proving may be a part of reasoning, reasoning generally refers to the process of developing arguments, which may not necessarily have a rigid logical structure.

Many mathematicians and mathematics educators have claimed that reasoning and proof are fundamental to doing and learning mathematics (Ball, Hoyles, Jahnke & Movshovitz-Hadar, 2002; Epp, 1998; Hanna, 2000, Herbst & Brach, 2006, Stylianides, 2009). However, despite the scholars' advocacy about the importance of such activities, students still have difficulties with reasoning and proof. Several studies documented secondary school students' difficulties with producing written proofs (e.g., Bell, 1976; Fujita & Jones, 2014; Harel & Sowder, 1998; Healy & Hoyles, 2000; Senk, 1985; TIMMS, 1998; Tinto, 1988). Several studies also show that mathematics textbooks have a significant influence on students' opportunities to learn reasoning and proof in secondary school (e.g., Fujita & Jones, 2014; Otten, Gilbertson, Males, & Clark 2014; Stylianides, 2009; Thompson & Senk, 2014).

In this study, I argue that studying textbooks is important because textbooks as artifacts of reform policies influence both teachers and students. This argument is demonstrated by the fact that the design of the curriculum in any particular country influences and molds students' mathematical reasoning and proof skills (Fujita & Jones, 2014; Healy & Hoyles, 1999; Otten et

al., 2014). Additionally, students experience the intended school mathematics curriculum (as specified in documents such as the national curriculum in England and Japanese "Course of Study," or even the national curriculum in Trinidad and Tobago) through the textbooks that are used in their classrooms (Schmidt et al., 2001; Valverde et al., 2002). Furthermore, it is worthwhile in any country to carefully examine the ways in which a process such as reasoning and proof, which is integral to what it means to do mathematics, is treated in the written curriculum. Thus, textbooks constitute an important component of the potentially implemented curriculum and it is worth examining in order to understand the implication of mathematics education reform in Trinidad and Tobago.

Students' Performance on Reasoning and Proof Items in CSEC

The recent reform policies in Trinidad and Tobago emphasize the importance of students engaging in reasoning and proof. According to policy documents, secondary school students should be given opportunities to engage in pattern identification, conjecturing, and formulating proof arguments, and non-proof arguments throughout their mathematical experiences (Trinidad and Tobago, 2005, 2009). The most recent Caribbean Secondary Education Certificate (CSEC)[5] mathematics syllabus also states that students should engage in the practice of constructing reasoned arguments and critiquing the reasoning of others. Underlying these recommendations is the assumption that when students engage in reasoning and proving, these experiences may lead to the development of deeper conceptual understanding of mathematical content (de Villiers, 1999; Dreyfus, 1999, Stylianides, 2009) and their appreciation of the purpose of reasoning and proof in mathematics (Lampert, 1990; NCTM, 2000).

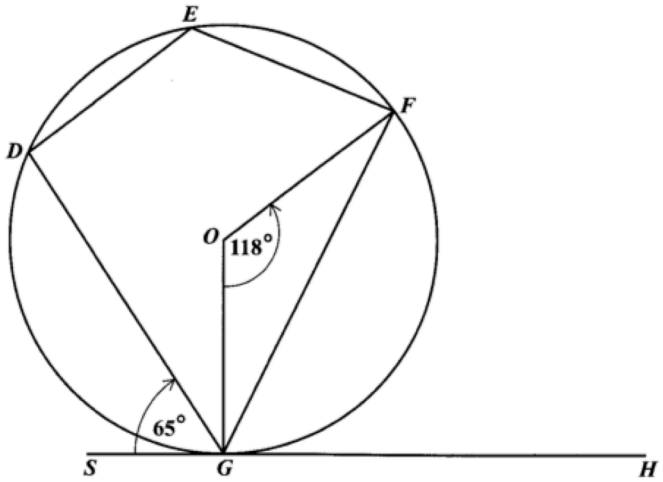
[5] The Caribbean Examination Council (CXC) grades the CSEC examinations. There are 31 CSEC subjects covering a range of purely academic and vocationally related subjects. A successful completion of CSEC examinations gives entry into post-secondary institutions in the Caribbean, Canada, the UK, and the United States.

Despite the proof-favoring policies in Trinidad and Tobago, students have shown difficulty with proof-based items in terminal assessments. Findings from an initial analysis of students' performance in the CSEC mathematics examinations indicate that students demonstrate difficulty with mathematics. For example, CXC's report on students' performance in the CSEC June 2013 mathematics examination showed that despite mathematics continuing to be the subject with the highest number of exam candidates, 102,214 only 35 % (approximately 35,775) passed the exam. This was a slight improvement from the 33% of students who passed the exam in 2012 (CXC Subject Award Committee, 2013). The CXC report also stated that students have difficulty with proof, particularly with questions asking students to explain why a solution or argument holds, and to make and prove conjectures. For example, in the CSEC January, 2014 exam, one of the reasoning and proof questions shown in Figure 2.1 required students to justify how they calculated the values of certain angles measures in a given diagram. Although the item did not explicitly require students to prove a result, the item did prompt students to engage in reasoning about the steps they would take to determine the values of missing angles. According to the examiners report, the question was attempted by 46% of candidates writing the exam. Of these 1% earned the maximum available marks (i.e. 5/5). The mean score was 0.78 out of 5. The report stated that overall the students' performance was unsatisfactory and although some students correctly calculated the missing values, they were unable to justify why their solution was correct. The examiners suggested in the January 2014 CXC Subject Award Committee's report:

> Teachers should encourage students to use mathematical terms to describe the relationship between angles formed when parallel lines are crossed by a transversal.

Candidates should be drilled in the practice of stating the reason or reasons for answers derived from the Geometry of plane figures. (p. 11)

10. (a) The diagram, below, **not drawn to scale**, shows a circle, centre, O. SGH is a tangent to the circle, $\angle FOG = 118°$ and $\angle DGS = 65°$.



Calculate, **giving reasons for EACH step of your answer**, the measure of:

(i) $\angle OGF$ **(2 marks)**

(ii) $\angle DEF$ **(3 marks)**

Figure 2.1: Example of a CSEC reasoning and proof question adapted from CSEC January 2014.

As a result of these findings about students' low performance on items in the CSEC examination, there exists a need to examine the quantity and quality of opportunities for students to develop reasoning and proving abilities. Furthermore, reasoning and proof in school mathematics does not only serve the role as providing opportunities for explaining and verifying

mathematical claims. Reasoning and proof also serves as a means for helping students develop valuable skills and attitudes in articulating their mathematical knowledge and providing formative assessment information for teachers (Staples, Bartlo, & Thanheiser, 2012). Reasoning and proof is considered a broad practice, encompassing identifying patterns, constructing and testing conjectures, developing non-proof arguments and constructing proofs. However, some researchers have claimed that students may not realize that informal mathematical experiences such as providing explanations to support their claims as being related to and could lead to constructing a proof. In the case of Trinidad and Tobago, students may not realize that opportunities such as the CSEC exam question in Figure 2.1 could support their development of reasoning and proof skills. Therefore, it is worthwhile examining whether such opportunities are evident in the textbooks students use in secondary school mathematics. Overall, the students' poor performance in reasoning and proof CSEC examination items coupled with the emphasis on reasoning and proof by policy documents in Trinidad Tobago initiates the need for an examination of the textbooks.

Research Questions

The research questions that guide this study are:

RQ.1 What is the nature and distribution of the opportunities for reasoning and proof within the Geometry sections of secondary school textbooks in Trinidad and Tobago?

RQ.2 How do the reasoning and proof opportunities in Geometry sections of the textbook expositions compare to the opportunities in student exercises?

The two-research questions are important to my inquiry because they provide a description of the nature of the proof-related opportunities in each textbook. The first research question explores how proof-related opportunities are distributed throughout the selected

Geometry topics. Additionally this question helps in determining whether the activities of pattern identification, conjecturing, developing non-proof arguments, and construction of proofs constitute the opportunities offered in the textbooks for students to reflect on and engage in reasoning and proof. Overall, the first research question helps in understanding the quality of the opportunities for students to learn reasoning and proof and the possible implications for students learning.

The second research question investigates how the various authors present proof in the Geometry sections of the textbook. Additionally this question explores whether students are continually offered reasoning and proof opportunities in both expository sections and exercises in the textbooks[6]. This helps to describe the coherency of opportunities for reasoning and proof presented throughout the textbooks in their respective expository sections and student exercises. The second question investigates how students are introduced to the processes of reasoning and proof in the expository sections of the textbooks. Moreover, the questions can help in determining how the opportunities for reasoning and proof afforded in the student exercises compare to the quality and quantity of the opportunities in the expository sections of each textbook. This comparison is important because it may help researchers understand the type of support available in textbooks for the teaching and learning of reasoning and proof. Overall, these research questions collectively provide insight into the type of reasoning and proof opportunities afforded in the written curriculum and also whether the extent of these opportunities is in agreement with the reform recommendations about reasoning and proof.

In this study, my inquiry is driven by the need to determine whether the current textbooks support the reform-oriented goals of increased opportunities for learning reasoning and proof.

[6] In Trinidad and Tobago, the recommended textbooks are typically organized to according textbook expositions, which introduces the definitions, theorems, and authors' demonstration of worked examples followed by the student exercises.

My quest is not to determine the best textbook for teaching secondary mathematics in Trinidad and Tobago. My main goal is to characterize the textbooks and see how they can shape instructional choices, and students opportunities to learn reasoning and proof. Furthermore by characterizing the various textbooks used in Trinidad and Tobago, researchers, policy makers, and education stakeholders can understand and resolve any existing issues with the written curriculum meeting the goals of the recent reform.

Context of Study

The Written Curriculum in Trinidad and Tobago

According to Stein et al. (2007) the written curriculum refers to the curriculum presented in policy documents, textbooks, or other teaching resources. In Trinidad and Tobago, the written mathematics curriculum consists of several documents, specifically, (a) an official government document describing the secondary school's mathematics program, (b) the terminal examination council's mathematics syllabus, (c) textbooks approved by the government, and (d) terminal examinations preparatory materials. These documents describe the epistemological consideration of curriculum reform, which refers to the knowledge, policy makers, and examining bodies consider necessary and suitable for the successful completion of secondary school (Schommer, 1994). These four requirements describe how the content in the curriculum should be organized and how students should acquire the content.

The official written curriculum document outlines six learning outcomes for all secondary students, (a) Aesthetic Expression, (b) Citizenship, (c) Communication, (d) Personal Development, (e) Problem Solving, and (f) Technological Competence. The goal of every core curriculum subject is the achievement of these essential learning outcomes by all students. Core curriculum subjects such as mathematics are expected to achieve this goal through, their content,

teaching, learning, and assessment strategies (Republic of Trinidad and Tobago Ministry of Education, 2009). The official curriculum document presents the content for learning and the expectations for teachers and students to achieve each of the aforementioned six learning objectives. This document also presents a suggested scheduled program of how the mathematics content areas should be covered for each grade level in secondary school.

The written curriculum also includes the CSEC mathematics syllabus, which contains the learning objectives for each content area that CXC assesses at the end of the fifth year of secondary school. Teachers are expected to use the mathematics syllabus to aid in preparing students for the assessment of mandatory and optional topics in the CSEC examination. The written curriculum also consists of the textbooks recommended by CXC. While the schools have the agency to choose their own textbooks, their selections are limited to the recommendations made by CXC and those offered by MOE Textbook Rental/Loan Program (TRP). The TRP provides textbooks and other educational resources to all students at no cost. Students are loaned the stipulated textbooks over the academic year of their respective grade levels. Students in forms, 4, 5, and 6 (ages 14 to 18) also receive CSEC syllabus and exam preparatory materials.

The final component of the written curriculum comprises preparatory materials such as past examination questions and answer booklets. CXC also provide teachers with guidelines about how each past examination questions are graded and suggestions for future preparation of students. The written curriculum in Trinidad and Tobago acts as a guide for assisting teachers and their student's master key information, ideas, and fundamental skills of the discipline of mathematics. The written curriculum also helps students move from novice to expert levels of performance in mathematics.

The written curriculum has many purposes. Textbooks and syllabus explain the body of mathematical knowledge that policy makers and examiners consider as appropriate for students. They also are powerful tools for teaching and learning. Textbooks, syllabi, and terminal examinations determine what is school mathematics (Dörfler & McLone, 1986). They also collectively serve as an accountability and *control function* (Woodward, 2004) meaning they act as a way of holding teachers and students accountable to the mathematical knowledge taught and learned in secondary school. Remillard (2000) claims that teachers have a tremendous mediation impact between the written curriculum and students' learning, and thus the connection between the written curriculum content, and students' achievement is not straightforward. Therefore it is worthwhile studying the extent and nature of the proof-related opportunities of the textbooks, syllabus, and examination materials in Trinidad and Tobago. Additionally, exploring how teachers interpret and use these opportunities to plan and enact classroom instruction.

Literature Review

In past studies examining opportunities for reasoning and proof the main tenet of the results is that most opportunities are offered in Geometry (Fujita & Jones, 2014; Hanna & de Bruyn, 1999; Otten et al., 2014). In those studies examining non-Geometry areas, number theory offered the most opportunities for reasoning and proof (Stylianides, 2009; Thompson et al., 2012). The common characteristics among most of these studies included: (a) the unit of analysis and (b) the conceptualization of the framework. The units of analysis for these studies were textbook expositions and student exercises. By textbook expositions, I refer to the narrative text containing introductions definitions, axioms, theorems, and worked examples. A student exercise refers to any task that provides an opportunity for students to use the content given in the expositions.

The conceptualization of the reasoning and proof used in the past studies was guided by either the Principles or Standards for School Mathematics (PSSM) or the Third International Mathematics Science Study (TIMSS). Stylianides' (2009) definition of reasoning and proving also informed the development of more recent frameworks in particular, the one I use for this study. In the next section I discuss the pertinent findings and features of past research on reasoning and proof in secondary school textbooks.

Research on Reasoning and Proof in Secondary School Textbooks

Overall, the range of research analyzing secondary school textbooks provide researchers with an understanding of the type of support teachers and students receive when teaching and learning reasoning and proof. The existing research also suggests three common findings about opportunities for reasoning and proof. Firstly, there exist huge variations in the percentages of reasoning and proof tasks from one textbook to another (Fujita & Jones, 2014; Hanna & de Bruyn, 1999; Otten et al., 2014). Even in comparisons of U.S. reform-based textbooks, researchers found significant differences in the percentage of tasks coded as reasoning and proving (Stylianides, 2009; Thompson et al., 2012). Secondly, in general, there are larger percentages of reasoning and proof opportunities in chapters specifically designed for writing proofs in Geometry. This finding is not surprising given Geometry is the area where students are officially introduced in writing proofs (Hanna & de Bruyn, 1999). Thirdly, students are offered varying levels of opportunities to engage in reasoning and proof across different content areas, with Geometry still being the prevalent topic for reasoning and proof.

In research on reasoning and proof in non-Geometry mathematics textbooks, Thompson, Senk, and Johnson (2012) examined both the exposition and the student tasks of 20 reform oriented and conventional textbooks published in the United States. in the areas of polynomials,

logarithms, and exponent properties. Their theoretical framework was based upon the U.S. reform document, PSSM (NCTM, 2000). This framework informed the conceptualization of reasoning and proving in the development of later frameworks for analyzing textbooks (e.g., Fujita & Jones, 2014; Otten et al., 2014). The framework classified reasoning and proving opportunities into the following categories, (a) making a conjecture, (b) investigating a conjecture, (c) developing an argument, (d) evaluating an argument, (e) developing counterexamples, and (f) correcting a mistake. Overall, they found three main categories of reasoning and proof in the student textbook exposition sections, (a) no justification, (b) justification with a specific example, and (c) justification with a general case. They reported that while approximately 50% of all the properties of the three topics were justified by some argument in the textbooks, there were insufficient opportunities for students to engage in reasoning and proof in the exercises. For example, in textbooks designed for an Algebra 1 course, they found that 3.4% of 2,838 tasks involved reasoning and proving. In the Algebra 2 course textbooks, 5.4% of 3,937 tasks involved reasoning and proof, and in the Pre-calculus course textbooks 7.7% of 2,967 tasks asked students to engage in some aspect of reasoning and proof. Altogether, students had limited opportunities to engage in reasoning and proof in textbooks within non-Geometry areas.

In another U.S. based study, Stylianides (2009) examined reasoning and proving in the number theory, Geometry, and algebra units of a U.S. middle school (ages 11–14) reform mathematics curriculum, Connected Mathematics Project (Lappan et al., 1998/2004). In this study of both Geometry and non-Geometry areas, Stylianides conceptualized the notion of reasoning and proof to design his theoretical framework. According to Stylianides (2009) a mathematician's work in the development of a proof is governed by activities such as identifying

patterns, making conjectures about these patterns, and developing arguments to show the validity of these conjectures. “These activities aid mathematicians in understanding mathematical relationships, building a foundation for the development of proofs” (p. 258). Therefore, Stylianides’ framework incorporates all the major activities that are involved in the acquisition of mathematical knowledge. This conceptualization of reasoning and proof guided the design of future frameworks including the one I use for my study. Stylianides used this conceptualization to examine the reasoning and proof opportunities in both the textbook expositions and student exercises. Stylianides’ main objective was to firstly identify the components reasoning and proof promoted in textbooks and secondly the purposes of these components in students’ engagement with reasoning and proof. For example, he looked at whether a pattern served as either a precursor or non-precursor to making a conjecture. The identification of these purposes helped in determining whether the reformers’ goal of allowing students to engage in all these activities of reasoning and proof are promoted by the reform oriented textbook Connected Mathematics Project curriculum (Lappan et al., 1998/2004).

As a result, Stylianides’ findings highlighted issues surrounding the role of reform-based textbooks promoting reform policies and the distribution of opportunities among different content areas. For example, Stylianides’ analysis suggested that about 40% of the 4,855 tasks that he examined involved reasoning and proof. Of these tasks, 62% were coded as rationales, which he described as informal arguments because they were missing some component such as references to previously proven theorems or definitions. Only 5% of the reasoning and proving tasks involved valid arguments. In addition, he found that there were more reasoning and proof opportunities in the number theory unit than in the Geometry and Algebra units. This finding contradicts the findings in other studies, which show that Geometry units usually have the

highest percentage of reasoning and proving opportunities (e.g., Hanna & de Bruyn, 1999; Otten et al., 2014). With regard to the elements of reasoning such as pattern identification, he found that a total of 18% of the reasoning and proof tasks provided opportunities for students to identify definite patterns, 6% of the tasks involved plausible patterns or patterns in which there was more than one possibility, and 12% of the tasks required a demonstrative argument meaning a formal proof. This study highlights the issue that even reform-oriented textbooks have a central role in promoting the goals of reform policies. In any reform movement textbooks are useful vehicles of change because they influence the content students receive and the decisions teachers make during instruction of the content (Cai & Cirillo, 2014; Stein, Remillard, & Smith, 2007). Stylianides concluded that in order to achieve the reform's objective that students in the U.S. must be given opportunities to engage in all activities of reasoning and proving in all content areas (NCTM, 2000), textbooks must reflect these ideologies and provide instructional support for teachers.

In non-U.S. territories, researchers also examined the reasoning and proving opportunities in textbooks in secondary school (e.g., Fujita & Jones, 2014; Hanna & de Bruyn, 1999; Nordström & Löfwall, 2006; Stacey &Vincent, 2009). For example, Fujita and Jones (2014) conducted an analysis of two Geometry chapters of one selected Grade 8 textbook used in Japan. They derived their theoretical framework from previous research with the Third International Mathematics Science Study (TIMSS). Fujita and Jones used an element of the TIMMS framework along with a conceptualization of reasoning and proving informed by the research of Stylianides (2009) and Thompson et al. (2012). However unlike these studies their unit of analysis was blocks of texts (narrative, worked examples, activities, exercises, and graphics) in the exercise and non-exercise sections of the chapters. Fujita and Jones used their framework to

characterize the way in which reasoning and proof is developed and introduced in the Geometry chapters of the textbooks. Their analysis indicated that most non-exercise blocks included opportunities to recall properties and theorems. Additionally in these non-exercise blocks, students were asked to make conjectures and engage in direct proof. In the exercise blocks, 100% of the exercises offered opportunities to engage in reasoning and proof. Thus these findings indicate that Geometry offers substantial number of opportunities for reasoning and proof in a non-integrated curriculum. However these results could possibly be different if this study analyzed additional chapters other than those designed for developing reasoning and proof skills.

In another international study of nine Australian eight-grade textbooks, Stacey and Vincent (2009) examined the modes of reasoning used in exposition sections relating to seven different topics appearing in the textbooks. Stacey and Vincent used the terminology "modes of reasoning" to describe the nature of reasoning presented in the expository sections of the textbooks. Although their focus was a range of issues including the procedural complexity of problems, they found an overall lack of problems requiring deductive reasoning. Stacey and Vincent found that the percentage of deductive explanations given varied by textbook and topic. For example, all of the explanations for the area of a trapezium (nine of nine) were deductive while only 17% (one of six) of the explanations for dividing fractions were deductive.

In another study, Nordström and Löfwall (2006) analyzed two Swedish secondary school mathematics textbooks. They found that although the textbooks offered opportunities for reasoning and proving, exercises using the word "prove" did not explicitly specify the nature of the expected argument student must produce. For example, some arguments could be formal narrative style proofs or some valid student responses could be informal, meaning they relied on

the effective use of a few examples. Hanna and de Bruyn (1999) examined students' opportunities to learn proof in two of the most popular textbooks designed for students in their final year of secondary school in Canada. They examined the following areas: Geometry, algebra, functions and relations, exponents and logarithms, and trigonometry in both the exposition sections and student exercises. Their analysis showed that over 80% of the proof opportunities were offered in the Geometry section, while only a few instances of proof occurred in the area of functions and relations and the topic exponents and logarithms.

In light of these findings, my study can potentially contribute to the literature by demonstrating how textbooks constitute an important component of analyzing the "potentially implemented curriculum" Cai and Cirillo (2014). Textbooks mediate between the intended opportunities for reasoning and proof and those actually implemented in classrooms. This study could help in understanding the possible links between the design of textbooks and development of students' reasoning and proof abilities. Additionally the results of this study along with classroom observations could potentially highlight broader issues associated with the international calls for the necessity of deductive reasoning and proof in all students' mathematical experiences.

Theoretical Framework

In this textbook analysis I use a framework developed by Otten et al. (2014), for the purpose of investigating the opportunities for reasoning and proof in secondary school mathematics textbooks. Otten and colleagues examined the mathematical nature of reasoning and proving opportunities in six U.S. secondary-level Geometry textbooks. Their analytical framework classified the type of mathematical statement, type of justification in the textbook exposition and student exercises, and the expected student activity in the student exercises.

Additionally, Otten and colleagues used the framework to investigate the type of statements and students exercises, which provide opportunities for students to engage in and reflect on reasoning and proof. The framework focused on the quantity and quality of opportunities for reasoning and proof. The latter is important in that the quality of opportunities could possibly help students see the intellectual need for reasoning and proving as a mathematical practice. Otten and colleagues used this framework to analyze the Geometry content in six textbooks. However, in Trinidad and Tobago, there is no separate Geometry course and all mathematics topics are integrated. Therefore, I examine six common selected Geometry topics in three textbooks. The use of this framework is useful in comparing textbooks in Trinidad and Tobago with other textbooks in the United States for which this analysis has been previously used. This comparison could provide an interesting discussion about how the nature of reasoning and proof opportunities in an integrated curriculum textbook compares with those offered in a non-integrated or purely Geometry textbook.

In their study, Otten et al. (2014) paid careful attention to the nature of the exposure and the mathematical context of the reasoning and proof activities in Geometry textbooks. Therefore, their study focused on students' "opportunities to learn" (Floden, 2002; Hiebert & Grouws, 2007; Husen, 1967) reasoning and proving in the secondary school mathematics textbooks. The framework used by Otten et al. is a modification of the analytic framework developed by Thompson et al. (2012) and is based on the NCTM's (2000) reasoning and proof process standard and the Third International Mathematics and Science Study (TIMSS). Figure 2.2 shows this analytical framework.

The theoretical framework that guides this study contains two dimensions indicated by the rows and columns (see Figure 2.2). The first dimension in the columns consists of the units of

analysis, namely the textbook expositions and student exercises. The student exercises are further sub-divided to reflect the nature of the expected student activity (i.e., activities related to mathematical claims and those related to mathematical arguments). I explain these further in a later section.

	Exposition	**Student Exercises**	
	Properties, Theorems, or Claims	Related to Mathematical Claims	Related to mathematical Arguments
Mathematical Statement or Situation	General Particular	• General • Particular • General with particular instantiation provided	• General • Particular • General with particular instantiation provided
Expected Student Activity		• Make a conjecture, refine a statement, or draw a conclusion • Fill in the blanks of a conjecture • Investigate a conjecture of statement • Perform a geometrical calculation with number and explanation (GCNE)	• Construct a proof • Develop a rationale or other non-proof argument • Outline a proof or construct a proof given an outline • Fill in the blanks of an argument or proof • Find a counterexample
Justification (Or environment for exploration)	• Deductive • Empirical • Outline • Past or Future • Left to student • None	• Deductive • Empirical • Implicit	• Deductive • Empirical • Implicit
Type of Opportunities about Reasoning and Proof	Statements about reasoning and proof • Statements about identifying patterns • Statements about making conjectures • Statements about providing non-proof arguments such as explanations and rationales • Statements about constructing proofs	Exercises about reasoning and proof • Exercises about identifying patterns • Exercises about making conjectures • Exercises about making non-proof arguments such as explanations and rationales • Exercises about constructing proofs	

Figure 2.2: Analytic Framework for reasoning and proof opportunities from Otten et al. (2014).

The second dimension consists of the four categories of codes for the textbook expositions and student exercises (see rows 1 to 4 in Figure 2.2). The first category describes the type of mathematical statements in the textbook expositions and student exercises. The second category

presents the codes for the nature of the expected student activities for each task in the student exercises. The next category contains the codes describing the type of justifications in both the textbook expositions and student exercises. The final category describes the classification of the opportunities about reasoning and proofs in the textbook expositions and student exercises.

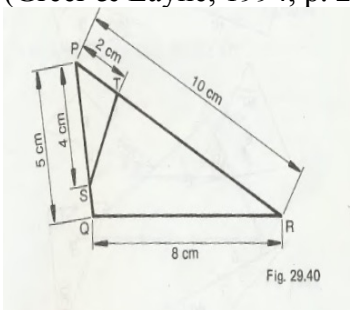
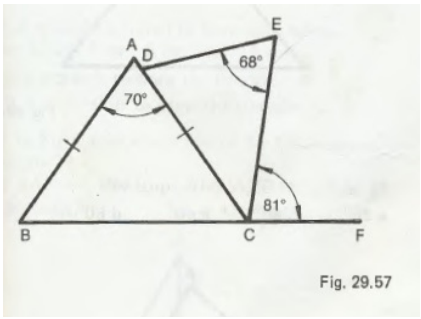
Classifying Types of Mathematical Statements

In their framework, Otten et al. (2014) classified the types of mathematical statements in the textbook expositions and student exercises. By mathematical statements, I refer to a mathematical claim about a single class or all classes of mathematical objects or situations. Otten and colleagues used the idea of the necessity principle (Harel & Tall, 1991) and the field of logic to provide a rationale for distinguishing between the types of mathematical statements. The necessity principle highlights the importance of students not only engaging in deductive reasoning but also appreciating the intellectual need for deduction in their mathematical experiences. This principle promotes reasoning and proving as an opportunity for students to understand underlying conceptual relationships, rather than as an arbitrary exercise imposed by an outside authority such as their teacher or textbook.

Otten et al. (2014) posited that deductive reasoning plays a pivotal role in justifying claims of all possible objects or situations under consideration (i.e. infinite classes). They captured this role of deductive reasoning by developing a category of codes relating to the mathematical statement or situation of reasoning and proving opportunities. The codes for mathematical statements are *general*, *particular*, and *general with particular instantiation provided*. In Table 2.1, I present examples of each code taken from the textbooks I analyzed in this study. I used these codes to classify the quantity and quality of mathematical statements promoting reasoning and proving.

Table 2.1

Coding of Mathematical Statements in the Textbook Expositions

Code	Description	Textbook Exposition Examples	Student Exercise Examples
General	A statement that concerns an entire class of objects or situations	When two parallel lines are cut by a traversal, the corresponding angles are equal (Greer & Layne, 1994).	Prove that all isosceles triangles have congruent base angles (Chandler, Smith, Ali, Layne & Mothersill, 2008).
Particular	A statement that concerns a specific mathematical object or situation	In Fig. 29.40 prove that Δs *PTS* and *PQR* are similar and calculate the length of *TS* (Greer & Layne, 1994, p. 215)  Fig. 29.40	In Figure 29.57 below $AB = AC$ BCF is a straight line. $\angle BAC = 70^0$, $\angle CED = 68^0$ and $\angle ECF = 81^0$. Prove that two of the sides of triangle *CDE* are equal (Greer & Layne, 1994, p. 219).  Fig. 29.57
General with particular instantiation provided	A statement that describes an entire class of objects but for which a specific member of the class has been indicated for students' use in reasoning.	NA	Consider an isosceles triangle *PQR* with a perpendicular bisector *OQ*. Prove that the bisector drawn from the apex angle of any isosceles triangle is perpendicular to the base (Toolsie, 2009, p. 457).

In my analysis, I define general mathematical statements as those statements that concern an entire class of mathematical objects or situations without exceptions. Particular statements refer to a statement that concerns a specific mathematical object or situation (or a finite set of objects). A general statement with particular instantiation concerns an entire class of mathematical objects but for which a specific member of the class has been selected for students' use in reasoning (Otten et al., 2014). This type of statement can be considered an exemplar or *a generic example* (Balacheff, 1988) of a class of objects or situation. The main purpose of this type of statement is to elucidate general characteristics of the entire class or situations under consideration. The focus in this case is not on the specific example but its use as a representative of a general class of objects. Therefore a student can use this exemplar or generic example to help them understand the general characteristics of an entire class of objects.

In this framework, Otten and colleagues also used the idea of universal generalization from the field of logic. This idea allows one to select an arbitrary element of a set to justify a claim about all elements in the set. Universal generalization allows one to draw any conclusions that do not appeal to specific characteristics of the selected element and extend them to the entire parent set (Hurley, 2006; Otten et al., 2014). For example, if one wishes to prove the general statement "*The sum of interior angles of any triangle is 180 degrees,*" one can begin the proof by stating "Let *PQR* be an arbitrary triangle." As a result of this idea, Otten and colleagues noted that the third code captures the fact that a preselected member of the set in question may accompany general statements in textbooks.

In such instances, students may not necessarily have to take the step of universal generalization themselves but can use the specific member provided in the textbook. The provision of a preselected member may be important if students do not notice that universal

generalization is possible as they reason with the specific selected member of the set. Overall, this code highlights a particular instance that satisfies the hypothesis stated in a general statement to be proven. In Geometry, this particular instance is usually in the form of diagram, which students subsequently use in their justification of the given general statement.

Within the framework, statement types and justification types are independent dimensions. This separation is due to the fact that general and particular statements can both be justified by empirical or deductive arguments. To highlight this difference, Otten and colleagues used the terms "general" and "particular" to refer only to statements and the terms "deductive" and "empirical" to refer only to justifications. In the same manner I use these terms as codes for statement and justification types respectively.

Classifying Justification Types in Textbook Expositions

The codes inherited for the justification types in the textbook expositions are: (a) deductive, (b) empirical, (c) justification left to the student, (d) past or future, and (e) no justification. Deductive justification refers to a logical argument, which uses definitions, postulates, or previously established results to support or prove a mathematical claim. In an empirical justification, the textbook provides a confirming example to a mathematical claim. Additionally an empirical justification may consist of a mathematical claim with accompanying diagrams. The sole purpose of the diagrams is for demonstrating examples of cases where the mathematical claim holds. In this case, the narrative text explicitly references the diagrams and highlights the purpose of the examples demonstrated by the diagram. When a justification is left to the student, the textbook explicitly states that students will be given the opportunity to prove the mathematical claim in a later exercise set. A justification is classified as past or future if the textbook explicitly refers to the justification presented in ether a previous or future chapter. The

final code, no justification refers to the case where the textbook does not provide any justification for a given mathematical claim.

Classifying Justification Types in the Student Exercises

In this framework the following codes were used for the type of justification that a student exercise required. In Table 2.2, I present examples of the codes inherited from Otten et al. (2014) for analyzing the justification types in the textbook exercises. In deductive justifications, the student exercises explicitly requests that students provide a "deductive argument" or a " logical chain" of justifications. An empirical justification requests that students provide measurements or confirming examples to solve a given task. In the final category, the student exercise requests that students engage in reasoning and proving (e.g., "Show…" or "Explain why…") but does not explicitly specify the nature of the argument to be produced. Otten and colleagues acknowledged that, with their definition of justification types, the majority of student exercises might fall in the implicit category. The inclusion of this code is built on the assumption that students may not necessarily interpret instructions to "prove," "justify," or "show" in the same manner that mathematicians or mathematics educators may interpret them. As a result, the code helps capture all of the possible actions students may produce when given these instructions. Furthermore, Otten and colleagues' inclusion of this code helps distinguish their framework as one focusing on opportunities for reasoning and proving in textbooks rather than students' reasoning.

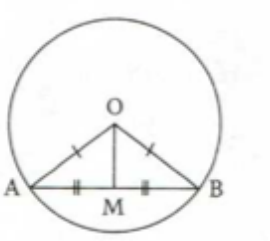
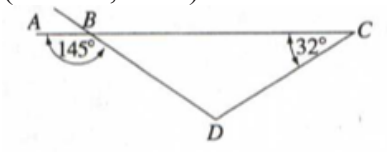
Expected Student Activity

In this framework, Otten and colleagues classified the expected student actions with respect to mathematical claims and constructing mathematical arguments in the student exercises. Using the work of Stylianides (2009), which defined the various activities involved in

reasoning and proving, they created the codes shown in row 2 of Figure 2.1, to ascertain the extent and nature of the opportunities for reasoning and proof offered to students. As a result of a preliminary analysis I conducted, I added a new code to the expected student activity related to mathematical claims. In the following section I introduce the new code.

Table 2.2:

Coding for Justification Types in Student Exercises

Code	Description	Student Exercise Examples
Deductive	The student exercise explicitly requests a 'deductive argument' or a 'logical chain of justifications'	In the diagram, *O* is the center of the circle and *M* is the midpoint of the chord *AB*. Prove that *OM* is perpendicular to *AB* (Greer & Layne, 1994) 
Empirical	The student exercise requests measurements or confirming examples.	Give a few examples of triangles that are congruent by (a) *SSS* postulate (b) *SAS* postulate (Chandler, Smith, Ali, Layne & Mothersill, 2008)
Implicit	The student exercise requests that students engage in reasoning and proof (e.g., "Show…" or "Explain why…") but does not explicitly specify the nature of the argument to be produced.	Evaluate angles *x* and *y*, giving reasons for your answers (Toolsie, 2009) 

Geometric Calculation with Number and Explanation

The new code I added to the framework is called "geometric calculation with number and explanation" (GCNE). This code is an extension of what previous scholars defined as a "geometric calculation with number" GCN (Ayres & Sweller, 1990; Hui-Yu & Silver, 2014; Küchemann & Hoyles, 2002). A GCN is a mathematical activity involving numerical calculations done on the basis of geometrical concepts, formulas or theorems. In a GCN, the request for an explanation of the steps in one's reasoning is not explicit but is implied as one may use geometrical concepts to obtain the solution. For example, a typical GCN task will request that students calculate the measure of a missing interior angle in a triangle given the measures of two other interior angles, say 30^0 and 50^0 respectively. In this activity, a student is expected to use the interior angle sum theorem for a triangle to calculate the missing angle. The student is not expected to explicitly state how the interior angle sum theorem supports their answer. A student can simply reference the theorem in parentheses nearby their algebraic calculations. The reasons supporting their calculations are not mandatory in their solution.

In a GCN, a diagram usually accompanies the given computational task (Hui-Yu & Silver, 2014). The purpose of the diagram is to help students visualize and understand the geometrical situation or object that will guide their reasoning. Based on my preliminary analysis, I define a geometric calculation with number and explanation GCNE as a student activity for reasoning and proof, which explicitly requires a geometrical computation and an accompanying reason, or explanation for the resulting calculation. As a result, students are expected to provide a non-proof argument justifying why their result is correct. The main difference between a GCN and a GCNE is that the GCN allows students to reason about a geometric situation using a

diagram while performing a computational task, whereas a GCNE goes even further to explicitly afford students the opportunity to provide a justification of the result of their calculation. The justification entails the student's provision of a non-proof argument to support their reasoning and computation. Figure 2.3 shows an example of a GCNE task in the Geometry textbooks I analyzed in this study.

.

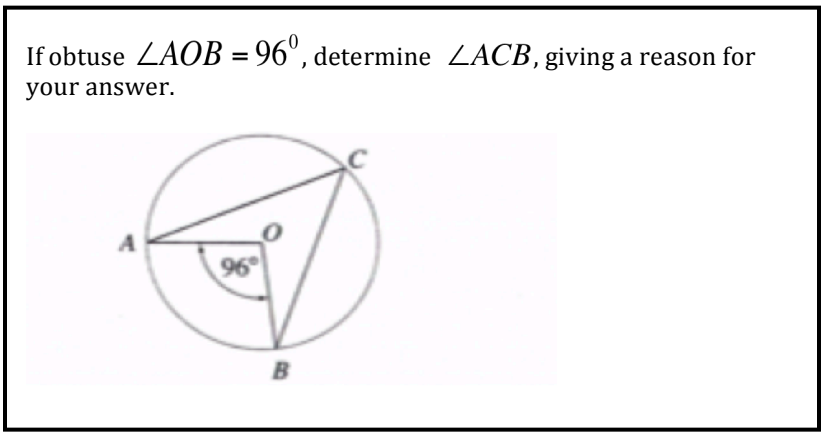


Figure 2.3: An example of a GCNE task. Adapted from "Mathematics a Complete Course" by R. Toolsie, p. 492.

As Figure 2.3 shows, the authors request that students find the measure of an angle at the circumference standing on the arc *AB*, given the angle at the center *AOB*, which stands on the same arc *AB.* In addition to calculating the measure of the angle, students are expected to provide a reason for the result of their calculations. Therefore students will be expected to use geometrical theorems about the angle properties of a circle as possible reasons or explanations for the result of their calculation. A possible theorem they may use will be that the measure of the central angle of a circle is twice the measure of the angle at the circumference subtending the same arc. As shown in Figure 2.3, the given angle *AOB* should be twice the measure of the requested angle *ACB*. Therefore, students will use this geometric result to help them explain why

ACB will be equal to half of the measure of *AOB* (i.e., $96/2 = 48^0$). Thus, the above task illustrates an example of an expected student activity I coded with the new category GCNE.

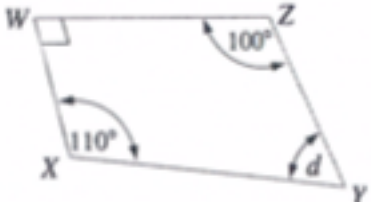
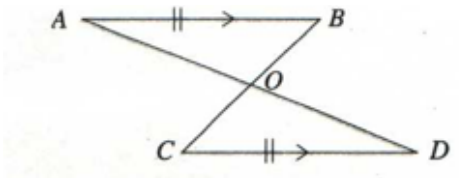
The expected student activities related to mathematical claims addresses the authors' intent for students to engage in pattern identification, conjecturing or developing a rationale during reasoning and proving. These activities help students move from inductive reasoning to deductive reasoning as they make generalizations of observed patterns and begin justifying their generalized claims. The aim of these activities is to refine students' abilities in constructing, testing, and critiquing conjectures. The expected student activities related to constructing mathematical arguments help students justify why a mathematical claim holds. These activities help students develop deductive reasoning skills as they write proof and non-proof arguments that explain their reasoning. With regard to the example presented in Figure 2.3, the expected student activity with respect to the mathematical claim would be to perform a GCNE task. This is indicated by the students' use of the geometrical claim on properties of angles intercepting on the same arc in a circle to calculate the missing angle *ACB*. The expected student activity related to arguments would be developing a rationale or non-proof argument. This is evident by the phrase "give a reason for your answer." This phrase suggests that students are expected to use the geometrical claims about angles in a circle to explain why they would perform a calculation a certain way to obtain how the requested measure of angle, *ACB*.

Statements About Reasoning and Proof

The final component of the analytic framework involves the classification of the opportunities for students to reflect on or think about reasoning and proof within the Geometry content of the textbooks. This is indicated by the fourth row of the framework (see Figure 2.2).

Table 2.3

Coding Categories for Student Exercises about Reasoning and Proof

Code	Description	Student Exercise
Pattern Identification	If a student exercise offers an opportunity for students to think about or engage in identifying a pattern.	*Construct about two to three quadrilaterals using a pencil and a ruler. Measure the size of the interior angles and find the sum for each quadrilateral. What do you observe?* (Toolsie, 2009)
Make a Conjecture	If a student exercise offers an opportunity for students to think about or engage in making a conjecture.	*Take a ruler and a pencil and draw your own triangles. Now take your protractor and measure each angle. After you have obtained the magnitudes of the three angles sum them. What do you observe?* (Toolsie, 2009)
Provide Non Proof Argument	If a student exercise offers an opportunity for students to explain with reasons about the validity of a mathematical claim. These explanations are not a formal proof just a rationale for why a statement holds	*Calculate the magnitude of angle d. Give reasons for your answer.* (Chandler, Smith, Ali, Layne & Mothersill, 2008) 
Construct a Proof	If a statement offers an opportunity for students to write up a proof or think about how to write a proof for a statement.	*In the figure, the straight lines AD and BC intersect at O in such a way that AB = CD and AB// CD. Prove that AO = DO and BO = CO. (*Greer & Layne, 1994) 

The respective codes for this category describe each of the mathematical activities in the processes of reasoning and proof defined in Stylianides (2009). Therefore, the statements and student exercises are categorized into one of the following activities: identifying patterns, making conjectures, providing non-proof arguments, or constructing a proof. Table 2.3 shows examples of the various categories for coding the student exercises about reasoning and proof.

In their framework, Otten et al. (2014) considered the four activities as those, which allow students to engage in or reflect on the mathematical practice of reasoning and proving. According to Otten et al. the inclusion of this classification is important in highlighting the extent to which each textbook promotes the necessity principle of reasoning and proof. By this they refer to the opportunities in the textbooks, which help students become aware of the intellectual need for proof and deductive reasoning in their mathematical experiences. Many students' difficulties with reasoning and proof involve the conception of these processes and their roles in mathematics (Hanna & de Bruyn, 1999; Otten et al., 2014). By reflecting on and engaging in these processes students are able to understand the role of these mathematical practices in creating and establishing mathematical knowledge (Hanna & de Bruyn, 1999). Therefore this final classification helps in understanding the quality and quantity of the opportunities for reasoning and proof embedded in the textbooks.

Methodology

Data Sources

As stated earlier, this study involves three contemporary textbooks designed for the preparation of students for Caribbean Secondary Education Certificate (CSEC) mathematics examination (see Table 2.4). Although schools have the agency to choose their own textbooks, the selections for this study were limited to the recommendations made by CXC and those

offered by MOE textbook rental program. The three selections corresponded to the Ministry of Education classification of a traditional textbook *Certificate Mathematics*, and a reform-oriented text *Mathematics a Complete Course* (Trinidad and Tobago, Ministry of Education, 2005). The second textbook replaced the first textbook as the only recommended textbook in the MOE textbook rental system (Trinidad and Tobago, Ministry of Education, 2006, 2007). The final book selection is a supplementary textbook recommended only by CXC because it reflects the recent changes in the CSEC mathematics syllabus.

Table 2.4

Textbooks Selected for Analysis

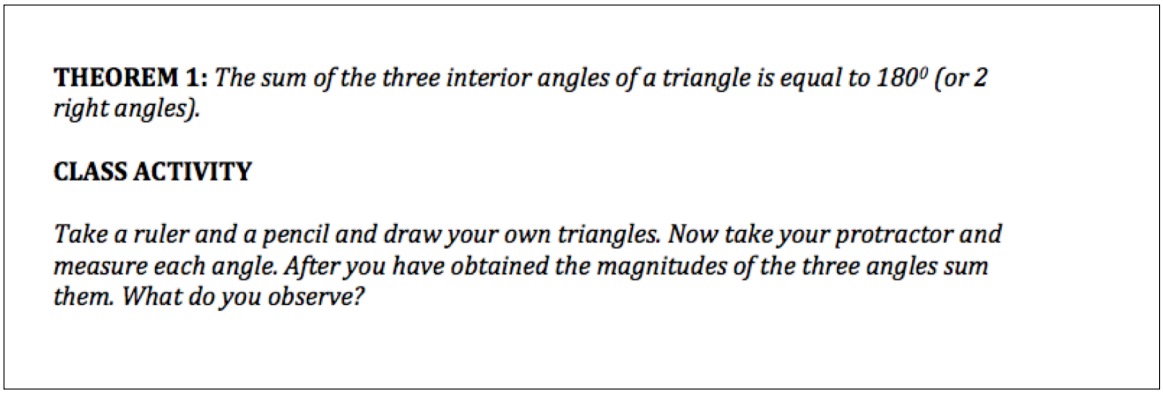
Title	Authors	Year
Certificate Mathematics (CM)	Greer, A. & Layne. C.	1994
Mathematics a Complete Course (MCC)	Toolsie, R.	2009
Mathematics for CSEC (MCSEC)	Chandler, S., Smith, E., Ali, F.W., Layne, C., & Mothersill, A.	2008

Each of the textbooks varies in their outlined goals for users as stated in their prefaces and introductions. However, each book's authors identify the common objective of providing students with practice CSEC type examination questions. In *Mathematics a Complete Course* (MCC) and *Mathematics for CSEC* (MCSEC), these questions are placed at the end of each topic examined by CSEC. For example, at the end of the Geometry sections of these texts there exists a review section containing past exam questions for students to practice. In *Certificate Mathematics* (CM) the exam type questions are presented at the end of topics grouped as mandatory and optional topics assessed in the CSEC examination. Geometry is one of the four topics that are grouped in the optional exam questions in this textbook.

The first textbook I selected for this analysis is *Certificate Mathematics* (CM*)*. Secondary Schools have used this book as one of the primary resources for the secondary mathematics in Trinidad and Tobago for the past thirty-two years. This textbook is designed as a revision (remedial) course in mathematics intended for those studying the General Proficiency Syllabus of CXC (Greer & Layne, 1994). The book's structure consists of both expositions and exercises. The expositions include introductory explanations, definitions, postulates, and worked examples. In my analysis the expositions and exercises are the units of analysis. Each chapter of the book ends with either a "Self-Test" or with a "Chapter Review" section of miscellaneous exercises. The exercises in the Self-Tests are all multiple-choice items. However, the Chapter Review exercises are short constructed response items. I include the chapter review in my analysis, because it contains the only available exercises that assess the Geometry content I analyze from this textbook. For example, in the case of the topic Similar Triangles, the Chapter Review contains the only student exercises assessing this topic. However the Self-Test items are not included in my analysis.

The second textbook I selected for this study is *Mathematics a Complete Course* (MCC). According to the author, this textbook is designed as both a self-paced course and supplementary teaching resource for the CSEC mathematics course. One of the books' objectives is to teach students to understand basic mathematical concepts and principles by placing greater emphasis on problem solving. The book offers introductory exercises, which assist students in exploring the basic definitions and theorems. The context of some of the problem-solving activities is based on everyday scenarios reflecting Trinidad and Tobago's culture. In some cases, historical facts are provided in the expository sections to clearly explain the contexts of the respective problem. In MCC, the expositions also consisted of suggested individual and group activities,

which is different from the other textbooks in my analysis. These class activities offer opportunities for students to investigate the validity of mathematical claims. For example, Figure 2.4 shows an activity, which offers an opportunity for students to explore the theorem about the sum of interior angles of a triangle. Firstly, the activity requires that students construct various triangles. Secondly, students are expected to measure the interior angles and find the sum of these angles. Finally, the activity requires that students identify a pattern concerning the sum of the interior angles. In my analysis, I included these activities in the coding of the textbook expositions.



THEOREM 1: *The sum of the three interior angles of a triangle is equal to 180^0 (or 2 right angles).*

CLASS ACTIVITY

Take a ruler and a pencil and draw your own triangles. Now take your protractor and measure each angle. After you have obtained the magnitudes of the three angles sum them. What do you observe?

Figure 2.4: An example of a class activity. Adapted from "Mathematics a Complete Course" by R. Toolsie, p. 443.

The third textbook I selected for my analysis is Mathematics for CSEC (MCSEC). This textbook is one of the recent textbooks suggested by CXC for the textbook rental program in Trinidad and Tobago**.** According to the authors, the book is tailored for students preparing for the CSEC mathematics examination. The book contains complete coverage of all topics outlined in the CSEC syllabus. A distinguishing feature of this textbook is the connection it makes to upper level studies in mathematics. For example, the authors suggest in some chapters (e.g., Vectors, Geometry, Sequences, and Statistics) the material covered provides the pre-requisites for the two-year CAPE advanced level mathematics courses in Form 6 (ages 16-18). The textbook also

includes comprehensive review tests after groups of chapters. The authors claim that this will assist students in revising the work covered in the corresponding chapters. The textbook concludes with three sets of multiple-choice exercises and three Model Examination papers. These are designed to test the entire CSEC syllabus. The sample examination items also include worked solutions.

Another distinguishing feature of this textbook is the inclusion of investigations of real-world scenarios within the textbook expositions. These investigations use previously introduced theorems, definitions, and postulates (see Figure 2.5). The example shown in Figure 2.5 is found in the Geometry section on Circles. In this investigation, students are expected to apply the mathematical theorems about angles in a circle from the previous sections in this textbook. This real-world scenario also utilizes ideas from previous topics in the algebra chapters about unit conversions with time. I included these types of investigations in Geometry in my analysis of student exercises.

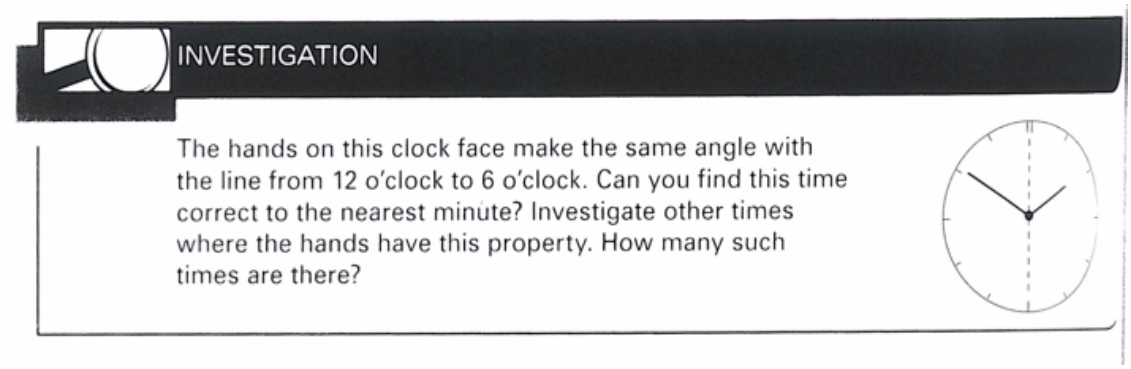


Figure 2.5: An example of an investigation activity. Adapted from "Mathematics for CSEC" by S. Chandler, E. Smith, F. W. Ali, C. E. Layne, and A. Mothersill, p.224.

Methods of Analysis

In each of the selected textbooks, I identified and examined all sections dealing with the geometrical properties of Triangles, Quadrilaterals, and Circles. The selection of these sections

was based on a preliminary analysis of the three textbooks. Table 2.5 shows the Geometry sections as they are sequenced in the three textbooks. Although each textbooks' sequence of the selected topics varied, I labeled and grouped the common content in the Geometry sections for my analysis in the following order (a) Triangles, (b) Congruent Triangles, (c) Similar Triangles, (d) Pythagoras' Theorem, (e) Quadrilaterals, and (f) Circles.

Table 2.5

Sequencing of Geometry Sections in the Textbooks

Textbook	Order of Topics in Geometry					
	Topic 1	Topic 2	Topic 3	Topic 4	Topic 5	Topic 6
CM	Triangles	Congruent Triangles	Similar Triangles	Pythagoras' Theorem	**Quadrilaterals**	Circles
MCC	Triangles	Congruent Triangles	Similar Triangles	Pythagoras' Theorem	**Quadrilaterals**	Circles
MCSEC	Triangles	**Quadrilaterals**	Congruent Triangles	Similar Triangles	Pythagoras' Theorem	Circles

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC. The bold font indicates the topic, which was not in the same position in authors' sequencing of topics in the respective textbooks.

As indicated in Table 2.5, the ordering of the topics in MCSEC was not in the same as the other two textbooks. Highlighting the topic Quadrilaterals in Table 2.5 shows the difference in the ordering of topics among the textbooks. In MCSEC, the authors opted to place the topic Quadrilaterals after Triangles before introducing congruency and similarity. The ordering of these topics is important because it determines the prior knowledge and resources students have for completing tasks. Therefore the proof of a mathematical claim may be different according to the knowledge students have acquired from previous textbook sections.

In each textbook the section on Triangles explicitly covered Geometry concepts associated with different types of triangles and angles. For example, in MCC, the section on Triangles as shown in Figure 2.6 below, defined the features of different triangles. As shown in Figure 2.6, the authors describe the distinguishing characteristics of right-angled, isosceles, and equilateral triangles. For instance, a right-angled triangle contains one angle with measure 90º, whereas an equilateral triangle has equal sides and all angles equal to 60º. This section did not cover any concepts related to congruency and similarity of triangles or the Pythagorean theorem. Therefore I considered these as different sections for my analysis. Because the curriculum in Trinidad and Tobago is integrated some non-Geometry topics were located within the Geometry sections. In those cases I omitted these topics from my analysis. For example, in MCSEC the topic of trigonometry was integrated into the sections covering Pythagoras' theorem. As a result, I omitted all mathematical statements and students' exercises within this topic so as to ensure uniformity of the Geometry sections I analyzed within the three textbooks.

Units of Analysis

In a similar manner to Otten et al. (2014), I included both textbook expositions and student exercises as my units of analysis. The reason for this is that the textbook expositions and student exercises contribute to the potential opportunities students have to engage with reasoning and proof. By textbook expositions, I refer to the paragraphs of narrative text in the body of the lesson presented and the sections or "text boxes" that contain formulas, definitions, theorems, or key conceptual ideas. Generally, textbook expositions refer to the sections of a lesson in which the textbooks' authors present information or ideas for the reader's consumption. Therefore, I included worked examples which are problems presented along with an explained solution as part of the textbook expositions. By student exercise, I considered an item for which students are

expected to actively engage in answering a question, solving a problem, or completing particular tasks. The unit of analysis for the student exercises was exercises as partitioned by numbers or letters in the textbook. For example, I considered an exercise with parts *a-e* as five separate exercises. A single exercise could receive multiple codes if it involved multiple reasoning and proof activities (e.g., investigate a conjecture and provide a rationale).

Triangles

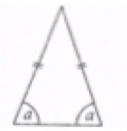
A **triangle** is a region of a plane enclosed by three straight lines.

The sum of the interior angles of a triangle is 180^0
An exterior angle is equal to the sum of the two interior opposite angles

There are three special triangles:

A **right-angled triangle** has one angle equal to 90^0

An **isosceles triangle** has two sides equal and the angles opposite those sides are equal.



An **equilateral triangle** has all three sides equal and all three angles are 60^0.

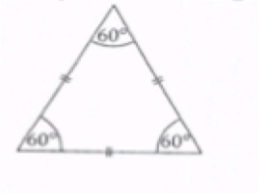


Figure 2.6: An excerpt of a section on Triangles from MCSEC, p.140.

Analysis of Textbook Expositions and Student Exercises

Within each of the six topics, I identified any expository mathematical statements and coded them according to the type of mathematical statement, justification type, and type of opportunities about reasoning and proof. The mathematical statements included theorems, postulates, geometric properties, formulas, identities, definitions, or other claims of mathematical truth (e.g., that a given diagram possessed a certain geometrical property). Within the expository sections of each topic, I analyzed sentences or paragraphs of text, which either (a) defined geometrical terms or concepts, (b) explained geometrical properties and accompanying diagrams, (c) demonstrated mathematical claims and properties in worked examples or activities, and (d) justified mathematical claims. In two of the textbooks, I included class activities and investigations about Geometry theorems and properties in my analysis.

Within each textbook, I analyzed and coded exercises that explicitly presented an opportunity for students to engage in reasoning and proof. For these opportunities, I included exercises, which directly asked students to prove a mathematical claim, investigate or make a conjecture, perform a geometrical calculation with number and explanation (GCNE) or justify a mathematical claim by developing a rationale or providing a non-proof argument. I did not include in my analysis exercises, which did not promote one of the aforementioned categories of reasoning and proof activity. For each student exercise qualifying as an example of promoting reasoning and proof, I coded the type of mathematical statement type and justification type.

To determine the statistical significance between the various textbooks during my analysis, I used Chi Square tests and hypothesis testing (i.e., *t*-test). For example, I used these test to investigate the statistical significance difference on the percentages of mathematical statement types, justification types, expected student activity, and statements about the practice

of reasoning and proof within and among the three textbooks. I used a level of significance of .05 for all statistical tests. For example, the statistical difference between textbooks for the percentage of mathematical statements type will be significant with a *p-value* less than or equal to .05.

Reliability Coding

To conduct reliability coding, a fellow researcher and I randomly selected one of the three textbooks. This was followed by our random selection of approximately 50% of my data from the textbook, which consisted of three sections from the selected textbook. We then individually coded the student textbook expositions and student exercises in the randomly selected sections. The resulting reliability for the textbook expositions was 96% for statement type and 94% for the justification type. The reliability check in the case of the student exercises yielded 92% agreement on statement type and 91% agreement on type of expected activity, and 98% agreement on justification type.

Results

In this section, I present findings of my analysis of the three textbooks I selected for this study. The findings are organized in sections answering the first research question about the nature and distribution of opportunities for reasoning and proof. Following this I answer my second research question comparing the opportunities for reasoning and proof in the textbook expositions and student exercises.

My first research question asked: *What is the nature and distribution of the opportunities for reasoning and proof within the Geometry sections of secondary school textbooks in Trinidad and Tobago?* In this study, I consider the nature of the opportunities for reasoning and proof in the textbooks to include the mathematical statement type and justification type. To answer this

research question, I first report findings about these two elements of the textbook expositions and student exercises. Another component of the nature of the reasoning and proof opportunities is the classification of the expected student opportunities in the student exercises. The description of this component is important because it helps to characterize the extent and type of opportunities for reasoning and proof offered in each textbook. As a result, I also report findings based on my analysis of the type of expected student activities in the student exercises. I also describe the statements and exercises about the practice of reasoning and proof according the to the classification in Stylianides (2009). This classification includes the four activities of (a) identifying patterns, (b) making conjectures, (c) developing non-proof arguments, and (d) constructing proofs. These descriptions will help in understanding the opportunities the authors of the textbooks afford for students to engage in and reflect on the mathematical practice of reasoning and proof. Finally, to understand the distribution of the reasoning and proof opportunities in Geometry, I organize the data by topics across the three textbooks. Throughout the results section, I will indicate the statistically significant differences measurement with the respective test statistic. I report the *t*-values in cases where I conduced a *t*-test and the equivalent test statistic for the cases where I conducted a Chi-Squared test. In the next sections I describe the mathematical statement and justification types in the textbook expositions and student exercises respectively.

Mathematical Statements Types in the Textbook Expositions

The textbook expositions of each textbook contained two types of mathematical statements, namely general and particular. The general statements included definitions, postulates, and theorems related to the six Geometry topics in the expository portions of each textbook. The other type of statements was particular and these described specific geometric

objects or scenarios. The particular statements were predominantly worked examples, which demonstrated the application of theorems and definitions to solve geometrical problems. Table 2.6 shows the types of mathematical statements that appeared related to reasoning and proof in the textbook expositions. The differences in the percentage of general statements and particular statements among the three textbooks were statistically significant different ($p < .05$). This is indicated by the asterisk in the respective columns for each textbook in Table 2.6.

Overall, general statements were prevalent in the expositions sections of the all three textbooks with over 75% of all statements in each textbook being general in nature. In MCSEC, 87% of the statements were general while only 13% were about particular geometrical objects or situations. In MCC, which had the highest number of mathematical statements, 79% of these were general, with about 21% being particular. Even in CM, in which the percentage of general statements was the lowest (77%), there were still more than three general statements for every one particular statement.

Table 2.6

Mathematical Statement Types in Textbook Expositions

Textbook	No. of Mathematical Statements	No. of General Statements (%)	No. of Particular Statements (%)	*t*
CM	56	43 (77)	13 (23)	4.01**
MCC	121	96 (79)	25 (21)	6.45**
MCSEC	61	53 (87)	8 (13)	5.76**

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC; * = $p < .05$; ** = $p < .01$

Mathematical Statements in Worked Examples

In each textbook, approximately 20% of all mathematical statements were worked examples. Table 2.7 shows the distribution of the worked examples among the three textbooks. I also present the type of justifications the authors used in the respective worked examples. In my analysis, all worked examples were coded as particular mathematical statement types because they represented specific geometric figures or situations. In MCC, the Chi-Squared test statistic, which measured the difference between the classifications of justifications the authors used, was statistically significant. In CM and MCEC the sample sizes were too small to accurately obtain a statistically significant test statistic.

Table 2.7

Distribution of Worked Examples in the Textbook Expositions

Textbook	No. of Worked Examples	No. of Justification type			
		Deductive (%)	Empirical (%)	Outline (%)	*Test Statistic*
CM	12 (21)	9 (75)	3 (25)	0 (0)	10.5
MCC	25 (20)	18 (72)	6 (24)	1 (4)	18.32**
MCSEC	13 (21)	10 (77)	1 (8)	2(15)	11.23

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC*; ** = $p < .01$*

The worked examples in the various textbooks demonstrated applications of theorems, postulates, and formulas introduced earlier in the textbook (see Figure 2.7). For example, Figure 2.7 shows a worked example of a proof about the congruency of a particular pair of triangles in MCC. In this example, the proof demonstrated the use of the Side-Angle-Side (SAS) postulate,

which the authors introduced earlier in the same expository section, to prove the congruency of the given triangles.

EXAMPLE
Given the triangles ABC and XYZ in Fig. 9.155, prove whether or not the two triangles are congruent.

SOLUTION

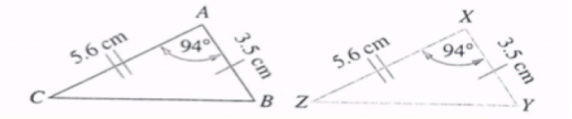


Fig. 9.155 Congruent triangles

Considering $\triangle s$ *ABC and XYZ*:
$AB = XY = 3.5$ cm (*corresponding sides equal*)
$\hat{A} = \hat{X} = 94°$ (*included angle equal*)
$AC = XZ = 5.6$ cm (*corresponding sides equal*).
Hence $\triangle ABC \equiv \triangle XYZ$ (*S.A.S.*).
So the *two triangles* are *congruent*.

Figure 2.7: A worked example using congruent triangles. Adapted from "Mathematics for CSEC" by S. Chandler, E. Smith, F. W. Ali, C. E. Layne, and A. Mothersill, p. 454.

This example demonstrates a deductive type of justification for proving the congruency of two triangles. In a later section, I will discuss the various justification types the authors used for the worked examples. Overall, the mathematical statement types in the textbook expositions were predominantly general in nature. More than three quarters of the statements were about an entire class of geometrical objects (e.g., inclusion of all triangles or quadrilaterals). In the next section I present results related to the type of justification in the textbook expositions.

Types of Justifications in Textbook Expositions

Within each of the selected topics, I coded the expository sections for the type of justification provided (see Table 2.8 and Table 2.9). In each textbook, the authors predominantly used empirical arguments to justify the mathematical statements. In Figure 2.8, I show an example of an empirical justification in one of the textbooks.

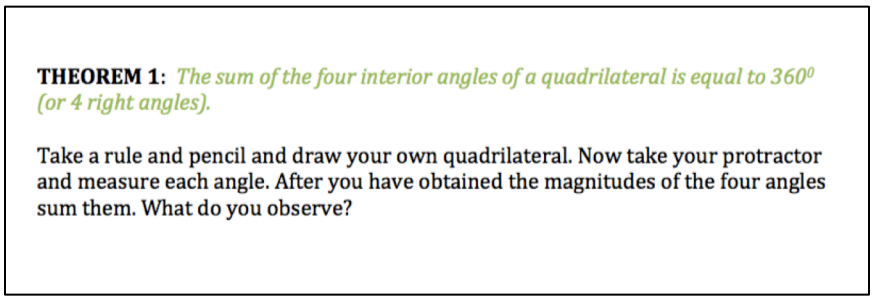


Figure 2.8: An empirical justification of a theorem. Adapted from "Mathematics a Complete Course" by R. Toolsie, p. 468.

In this example the author presented a theorem about the sum of interior angles of a quadrilateral. To prove this result the author suggested that students construct any quadrilateral. When the author stated, "take your protractor and measure each angle," he suggested that students use empirical measurements to obtain the interior angles. The author also suggested that students find the sum of the four interior angles they obtained through measuring. When the author asked, "What do you observe?" he seemed to prompt students to observe that the claim in the given theorem holds for the quadrilateral students constructed. This example demonstrates a case where the author used measurements and student-generated examples to justify a mathematical result. This was the only justification of the given theorem the authors provided in this textbook. The aforementioned example represents a case of an empirical justification.

In MCC, which had the highest number of mathematical statements, the author provided empirical justifications for a little over a half (52%) of the statements in the textbook expositions. In CM and MCSEC approximately 41% and 34% of the statements were justified by confirming examples of a few cases, which demonstrated a geometrical result or property presented in the respective textbook expositions.

Table 2.8

Deductive and Empirical Justification Types in the Textbook Expositions

Textbook	No. of Justification types (%)		
	Deductive (%)	Empirical (%)	*Test Statistic*
CM	12 (21)	23 (41)	-1.86*
MCC	33 (27)	63 (52)	-3.06**
MCSEC	18 (30)	21 (34)	-0.48

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC; * = $p < .05$; ** = $p < .01$.

The authors of the various textbooks also utilized deductive arguments to justify the mathematical statements. By a deductive argument, I mean the authors' use of logical chains of reasoning from the hypotheses to the conclusion to construct an argument that supports the validity of a mathematical claim or theorem. A deductive argument does not include the use of measurements or a few generated examples. I considered a deductive argument to be general in nature. In MCSEC, there was a somewhat even split between the percentage of deductive and empirical justifications. About 30% of the expository statements were justified by deductive arguments, whereas 34% were justified by empirical arguments. There was no statistically significant difference between these percentages for MSEC (See Table 2.8). In contrast to this finding, there was a statistically significant difference between the percentages for the empirical and deductive justifications provided in CM and MCC. In those two textbooks, there were about half as many deductive justifications as empirical justifications, with 21% and 27% deductive justifications in CM and MCC, respectively.

Other Justification Types in the Textbook Expositions

In the three textbooks, there was variation in the type of justifications provided by the authors. Table 2.9 shows the other type of justifications beyond empirical and deductive, the various authors used in the textbooks. Less than 10% of the statements' justifications were either left to the student as an exercise, justified in past or future chapters or as an outline of a proof argument. Among the three textbooks, there were only two occurrences of the left to student-type justification.

Table 2.9

Other Justification Types in the Textbook Expositions

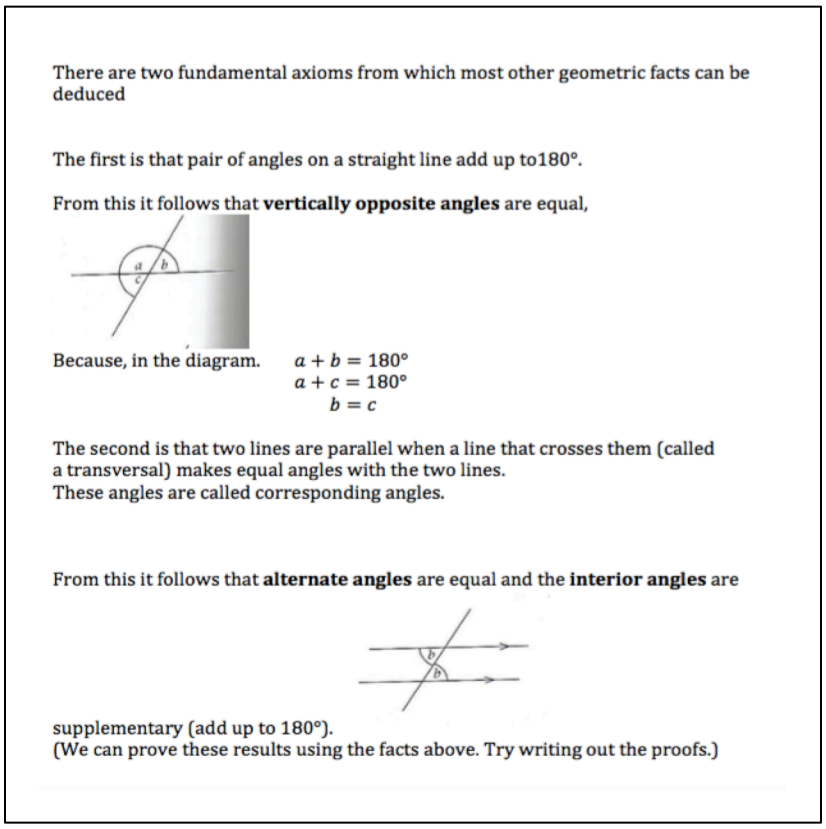
Textbook	No. of Justification types (%)				
	Left To Student (%)	Past Or Future (%)	Outline (%)	No Justification (%)	*Test Statistic*
CM	0 (0)	0 (0)	2 (4)	19 (34)	48.52**
MCC	0 (0)	2 (1.6)	7 (6)	16 (13.4)	24.44**
MCSEC	2 (3)	3 (5)	3 (5)	15 (24)	22.01**

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC; * = $p < .05$; ** = $p < .01$

The left to student type justification was different to a student exercise because of (a) its location in the textbook (b) the explicit requirement of the authors of the textbook. This type of justification would only be located in the textbook expositions. The authors would explicitly suggest that students should do the expected justification of a result or theorem as a practice exercise. The suggested justification would not appear as an actual exercise in the student exercises at the end of the respective section or chapter in the textbook. As a result, the suggested

justification would be the only opportunity for students to prove the theorem. Among the three textbooks, MCSEC contained both instances of this type of justification, accounting for 3% of the justifications the authors provided in this textbook.

Figure 2.9 shows an example of the left to student justification type. In this example, the narrative text was located at the beginning of the Geometry section in MCSEC. In this narrative, the authors introduce two fundamental axioms they consider useful for deducing other geometric facts in the subsequent Geometry sub-sections.



There are two fundamental axioms from which most other geometric facts can be deduced

The first is that pair of angles on a straight line add up to180°.

From this it follows that **vertically opposite angles** are equal,

Because, in the diagram. $a + b = 180°$
$a + c = 180°$
$b = c$

The second is that two lines are parallel when a line that crosses them (called a transversal) makes equal angles with the two lines.
These angles are called corresponding angles.

From this it follows that **alternate angles** are equal and the **interior angles** are

supplementary (add up to 180°).
(We can prove these results using the facts above. Try writing out the proofs.)

Figure 2.9: An example of left to student justification. Adapted from "Mathematics for CSEC" by S. Chandler, E. Smith, F. W. Ali, C. E. Layne, and A. Mothersill, p. 140.

The authors establish the first axiom "A pair of angles on a straight line add up to 180º." The authors use this axiom to prove that vertically opposite angles are equal (see proof in Figure 2.9). This proof is followed by the second axiom; "Two lines are parallel when a line that crosses

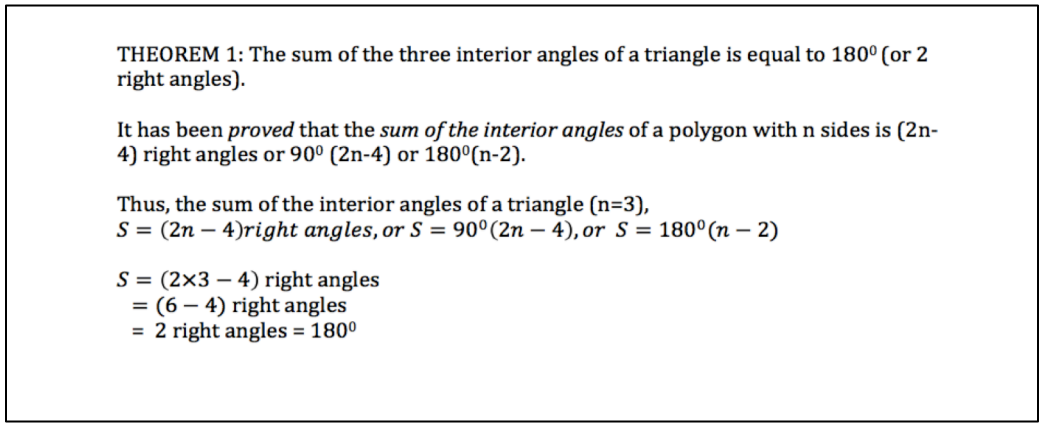
them (called a transversal) makes equal angles with the two lines[7].” The authors then introduced two other geometrical theorems, (a) “alternate angles are equal” (angles labeled as “b” in the second diagram in the narrative text of Figure 2.9) and (b) “interior angles are supplementary[8].” When the authors provided the hint, “We can prove these results using the facts above,” they seemed to explicitly reference the usefulness of the two previously established fundamental axioms for justifying the new facts. The phrase “Try writing out the proofs,” indicates that the justification of the two facts is left to the student as an exercise. This phrase also indicates that students are expected to use the hint the authors provided to prove these results.

Another type of justification that the authors used in the textbook exposition was “past or future.” MCC and MCSEC were the only textbooks containing this type of justification with 1.6% and 5% of the justifications, respectively. In this type of justification the authors would reference a justification of a mathematical statement in a past or future chapter. Figure 2.10 shows a typical past or future type of justification. In this example the authors provides a justification for the theorem “The sum of interior angles of a triangle is 180 degrees.” In providing a justification of this statement, the authors reference a proof of the result “The sum of interior angles of a polygon with *n* sides is *(2n-4)* right angles or 90^0 *(2n-4)* or 180^0 *(n-2)*.” The phrase “It has been proved” indicates the authors’ reference to an earlier proven argument, which I located in an earlier chapter introducing properties of polygons. Although the referenced proof holds for a general case of any polygon, the author suggests that the proof is applicable to a

[7] The definition of the corresponding angles formed by the intersection of two parallel can be made clearer to the reader with a diagram indicating which intersection in the figure affords the formation of the angles the authors are referencing in their definition.

[8] The following results “alternate angles are equal” and “interior angles are supplementary” in the MCSEC textbook refers to the definition of alternate interior angles, which is typically found in US Geometry textbooks.

triangle in the particular case of $n = 3$. By substituting this value of n into the general formula, the author shows algebraically that the sum of the interior angles of the triangle is equivalent to 180^0 or *2* right angles. This example is important, because it is representative of the cases of past or future justification in MCC and MCSEC. Only two textbooks provided past or future type of justifications. In both textbooks, the authors did not explicitly state where the result was or will be proven. For each textbook, I searched the preceding chapters and subsequent chapters of the textbook to find the proof referenced by the author when using past or future justifications. In all of these cases, the proof was located within the Geometry sections of the respective textbooks. In the example provided, the author did demonstrate the case of a specific polygon a triangle, when the variable n is equal to 3 in one of the general formulas. The author's proof of this case is based on the proof of the general formula in an earlier chapter of the textbook. The author's justification is also based on a past deductive justification of the general formula.



THEOREM 1: The sum of the three interior angles of a triangle is equal to 180^0 (or 2 right angles).

It has been *proved* that the *sum of the interior angles* of a polygon with n sides is (2n-4) right angles or 90^0 (2n-4) or 180^0(n-2).

Thus, the sum of the interior angles of a triangle (n=3),
$S = (2n-4) right\ angles, or\ S = 90^0(2n-4), or\ S = 180^0(n-2)$

$S = (2\times3-4)$ right angles
$= (6-4)$ right angles
$=$ 2 right angles $= 180^0$

Figure 2.10: An example of past or future justification type. Adapted from "Mathematics a Complete Course" by R. Toolsie, p. 443

In the textbook expositions, the authors also provided an outline of an argument as a justification of a mathematical statement or worked example. The authors presented the outlines as either (a) commentaries of how to proceed to prove a mathematical statement, (b) incomplete sentences of the proof for a mathematical statement, or (c) suggested steps for a proof of a

worked example. The example presented in Figure 2.9 also represents an outline type of justification. This example was the only case where I coded the textbook exposition justification as both an outline and a left to student justification type. As a result in MCSEC, the left to student exercises were also coded as outline justifications.

In the case of worked examples, the authors partially demonstrated the suggested argument in the outline. Figure 2.11 presents an example of an outline justification type for a worked example. In this worked example, although the authors provide the required information for justifying the theorem, they seem to offer an opportunity for students to use previously introduced postulates on congruency. Students are required to prove that *OM* shown in the given diagram is perpendicular to the side *AB* in triangle *AOB*. To construct a suitable proof for this worked example the authors provided an outline of an argument that will prove that *OM* is indeed perpendicular to *AB*.

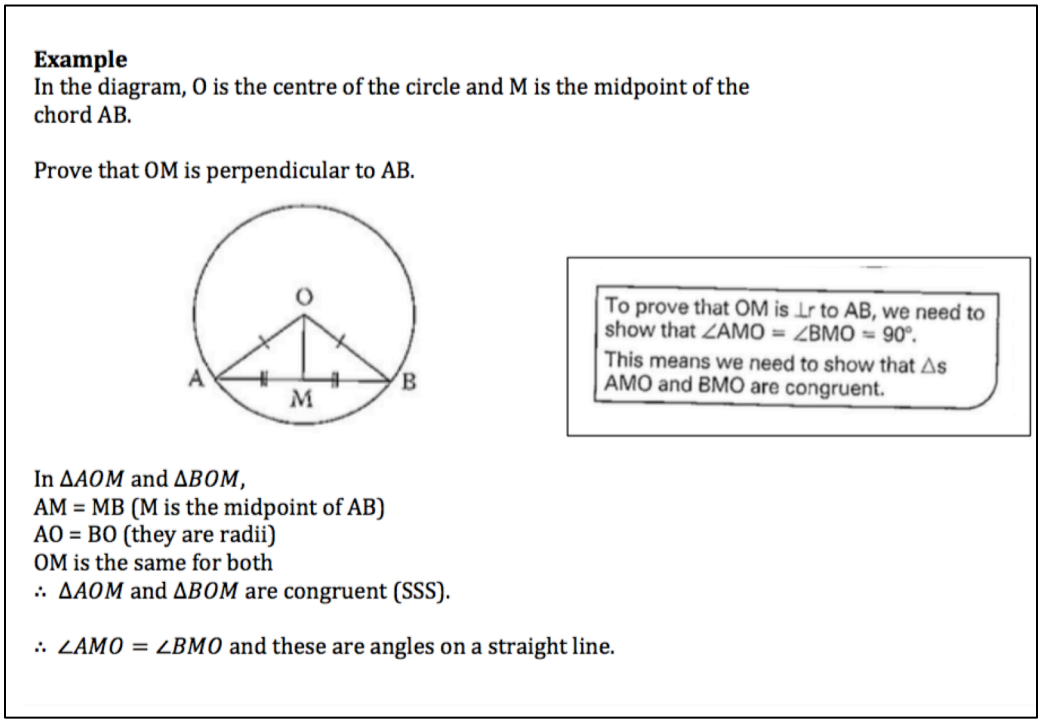


Figure 2.11: An example of an outline justification. Adapted from "Mathematics for CSEC" by S. Chandler, E. Smith, F. W. Ali, C. E. Layne, and A. Mothersill, p. 149

As shown in Figure 2.11 the authors suggested that to prove *OM* is perpendicular to *AB*, it must be shown that (a) angle *AMO* is equal to angle *BMO* which is equal to 90 degrees and (b) this would be possible by first showing that triangles *AMO* and *BMO* are congruent. With these two suggestions, the authors proceed to establish the second claim by application of the Side-Side-Side (SSS) congruency postulate. This example is also interesting in that the authors are using an outline justification followed by a demonstration of the deductive argument given in the outline. This example is representative of a worked example with an outline justification.

Among the three textbooks, there were cases in which no justification was provided for expository statements. MCC having the highest number of expository statements among the three textbooks had the lowest percentage of statements without any justifications. CM had the highest occurrence of statements not being justified (34%), whereas MCC had the least (13.4%).

Overall, MCSEC was the only textbook that employed all categories of justifications coded in the analysis. For example, MCSEC was the only book that the authors also left the justification of statements as an exercise for the students to complete (see Table 2.9). CM had the least variability in the type of justification provided for expository statements. The authors of this textbook only used empirical, deductive, and outline justification types. The percentage of outline justifications in this textbook was below 5% whereas the empirical and deductive justifications were used for 41% and 21% of the statements, respectively. On the other hand, MCC employed all justification types except left to student justifications (see row 2 Table 2.8 and Table 2.9). In MCC, empirical arguments were provided the most frequently and did not leave any justifications as an exercise for the students. However, this textbook provided outlines of possible proofs for theorems more frequently (i.e., 6% of the justification types) than the other two textbooks. The respective percentages of justifications within this category in CM and

MCSEC were 4%, and 3% respectively. Therefore the opportunity for students to complete the justification of a geometrical result using an outline was atypical among the three textbooks.

With regard to the worked examples, Table 2.7 shows that the authors predominantly justified these by deductive arguments. In each of the textbooks, the authors used deductive arguments for over 70% of their justifications for worked examples. In MCC and MCSEC, there existed a variation of other justification types of the worked examples. Specifically, 4% and 15% of the worked examples were justified by outlines. In MCC, over 20% were justified by empirical arguments whereas less than 10% in MCSEC were justified by empirical arguments. In CM, the authors did not use outlines to justify the worked examples. About a quarter of the worked examples were justified by empirical arguments with the remainder (75%) justified by deductive arguments.

Summary

Overall, the mathematical statements in the textbook expositions were predominantly justified by empirical arguments. Approximately almost half of all the statements' justifications used measurements or confirming examples of the applications theorems and definitions to construct empirical arguments. Of the other non-empirical justifications, deductive arguments provided justifications for mainly the worked examples and some mathematical statements. These deductive arguments accounted for a little over 25% of all the mathematical statements. Finally, the other justification types of past or future, left to the student, and outline accounted to less than 10 percent of the justifications in the textbook expositions.

Opportunities for Reasoning and Proof in the Student Exercises

Table 2.10 presents the number of student exercises providing opportunities for reasoning and proof in each textbook. Overall, I analyzed 519 student exercises combined from the three

textbooks. Among the textbooks, there was a statistical significant difference between the percentage of reasoning and proof and non-reasoning and proof in the student exercises.

Table 2.10

Opportunities for Reasoning and Proof in the Student Exercises

Textbook	No. of Exercises	No. of Reasoning and Proof Exercises (%)	No. of Non-Reasoning and Proof Exercises (%)	*t*
CM	110	39 (35)	71 (65)	-2.31*
MCC	241	185 (77)	56 (23)	8.31**
MCSEC	168	54 (32)	114 (68)	-4.63**
TOTAL	519	278 (54)	241 (46)	

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC; * = $p < .05$; ** = $p < .01$

Of all the student exercises within the three textbooks, approximately 54% offered opportunities for reasoning and proof. However two of the textbooks, CM and MCSEC had less than 40% of their respective exercises offering reasoning and proof-related opportunities. Both of these texts had over 60% of their student exercises having no opportunity for reasoning and proof (see Table 2.10). In CM and MCSEC, 35% and 32% of their respective student exercises offered opportunities for reasoning and proof. However, MCC, the textbook with the highest number of student exercises, had highest percentage of opportunities for reasoning and proof among all three textbooks (approximately 77%).

Mathematical Statements Types in the Student Exercises

As Table 2.11 shows, particular statements were prevalent in the student exercises of all three textbooks. In MCC, 85% of reasoning and proof exercises involved a particular statement. There was a statistical significant difference among various statement types for each textbook.

Table 2.11

Mathematical Statement Types in the Student Exercises

Textbook	No. of General Statements (%)	No. of Particular Statements (%)	No. of General with Particular Instantiation	*Test Statistic*
CM	10 (26)	21 (54)	8 (20)	7.54*
MCC	5 (3)	157 (85)	23 (12)	223.69**
MCSEC	2 (4)	40 (74)	12 (22)	43.11**
TOTAL	17	218	43	

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC; * = $p < .05$; ** = $p < .01$

Figure 2.12 shows an example of an exercise taken from the section on Circles. In this exercise, students are given a specific measurement of the geometric figure provided. In this case, the measure of angle *PQR* (i.e., 105 degrees) in the quadrilateral *PQRS*, which is inscribed in the circle. In the mathematical statement of this exercise, the specific measurement of the angle pertains to the particular geometric situation of an inscribed quadrilateral having one of its angle measurements as 105 degrees.

In the cyclic quadrilateral *PQRS*, angle *PQR* = 105^0. Evaluate angle RST, giving a reason for your answer.

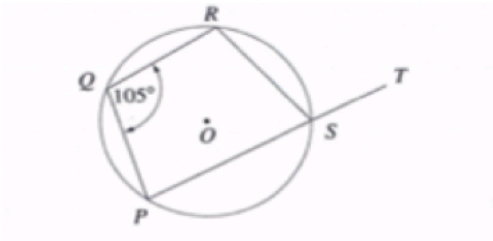


Figure 2.12: An example of a particular GCNE student exercise. Adapted from "Mathematics a Complete Course" by R. Toolsie, p. 494.

In this exercise, students are required to calculate the measure of angle *RST*, with the expectation that students use geometric theorems about angle measures in a circle to help them provide reasons for their calculated result for the angle *RST*. Therefore, the solution to this exercise will apply only to this particular situation or figure. This exercise is representative of the particular-type statements in a student exercise.

In MCSEC and CM, particular statements were occurred in 74% and 54% respectively in the student exercises (see Table 2.11). Although CM included the lowest proportion of particular statements in the student exercises among all three textbooks, it included the greatest proportion (approximately 26%) of general statements. The three textbooks had 12% to 22% of their exercises involving reasoning and proof around a general statement with particular instantiation. For example, in MCSEC, which had the greatest proportion of such statement type (22%), the student exercises required that students prove a mathematical claim by focusing on a selected particular case representative of a general class of objects.

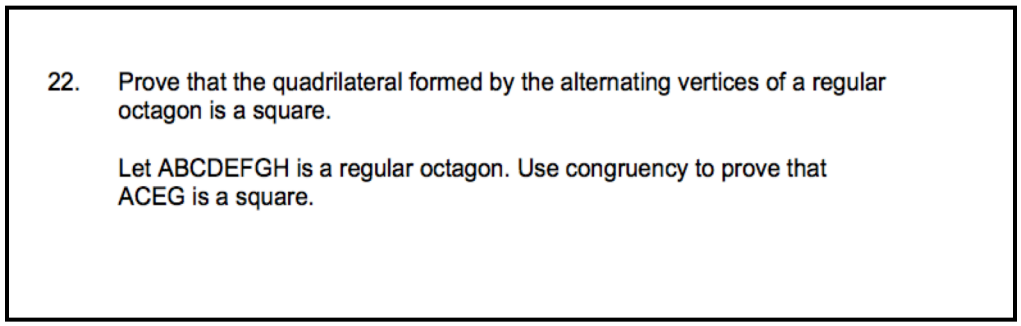


Figure 2.13: An example of a general with particular instantiation exercise. Adapted from "Mathematics for CSEC" by S. Chandler, E. Smith, F. W. Ali, C. E. Layne, and A. Mothersill, p. 150.

As shown in the example given in Figure 2.13, the student exercise asked students to prove the general result about the type of quadrilateral formed by the alternating vertices of a regular octagon. The question then, further specified by selecting a particular case of a regular octagon with vertices *ABCDEFGH* to prove the result. The author's use of this example demonstrates that students are expected to reason about the general result and construct a proof based on congruency theorems as indicated in the hint.

In MCC and MCSEC students were given more opportunities to reason with general cases with particular instantiation than purely general cases, approximately 9% and 18% more than general cases respectively. However, CM had 6% more general exercises than general with particular instantiation. Overall, most reasoning and proof exercises offered particular mathematical statements rather than general. Of the general statements, a greater proportion had particular instantiations provided for student's reasoning.

Expected Student Activity in the Student Exercises

As part of the analytical framework for this study, I analyzed the expected student activity in the student exercises. This component addresses two components of the nature of reasoning and proof, (a) the expected student activity about mathematical claims, and (b) the expected student activity about mathematical arguments. As stated earlier in the methodology

section, the expected student activities related to mathematical claims addresses pattern identification and conjecturing during reasoning and proving. Whereas, the expected student activities related to constructing mathematical arguments help students justify why a mathematical claim holds. These activities help students develop deductive reasoning skills as they write proof and non-proof arguments that explain their reasoning. These two components are useful in helping us further understand the extent of the type of opportunities for reasoning and proof afforded within the student exercises of the textbook. Table 2.12 shows the distribution of the various types of expected student activities related to mathematical claims.

Table 2.12

The Expected Student Activities Related To Mathematical Claims

Textbook	No. of Expected Student Activity in Student Exercises Related to Mathematical Claims				
	MakeConj (%)	InvestConj (%)	FillBlank (%)	CalcExp (%)	*Test Statistic*
CM	11 (28)	28 (72)	0 (0)	0 (0)	5.31*
MCC	12 (7)	38 (20)	0 (0)	135 (73)	8.23**
MCSEC	1(2)	30 (56)	5 (9)	18 (33)	4.56*
Total	24	96	5	153	

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC; *MakeConj = Make a conjecture; InvestConj = Investigate a conjecture; FillBlank =Fill in the blanks; CalcExp = Geometric Calculation with Number and Explanation (GCNE) or Calculate and explain; * = p < .05; ** = p < .01*

In two of the textbooks, the highest percentage of expected student activity related to mathematical claims was investigating a conjecture (InvestConj), with 72% and 56% in CM and MCSEC respectively. However, MCC contained the lowest percentage of exercises requiring that students investigate a conjecture at 20%. Exercises in which students were expected to

investigate a conjecture would typically contain a mathematical claim and require that students show or investigate its validity. Figure 2.14 shows an example of an exercise coded in this category.

In this example, students are given specific information about the various components of two triangles in the provided figure. The exercise requires students to investigate the conjecture that the given triangles *ABC* and *EBD* are similar. As a result, students are expected to investigate the validity of the conjecture by showing that the two triangles are indeed similar. Students can prove the similarity of the two triangles by showing that the corresponding sides of the pair of triangles are in the same ratio.

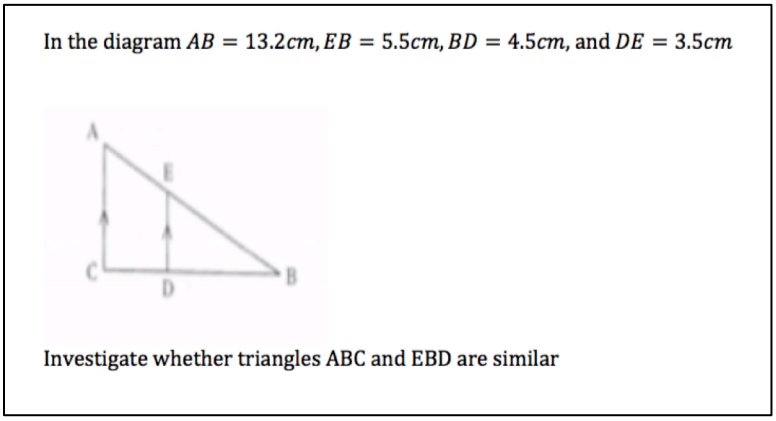


Figure 2.14: An investigate conjecture type of expected student activity. Adapted from "Mathematics for CSEC" by S. Chandler, E. Smith, F. W. Ali, C. E. Layne, and A. Mothersill, p. 147.

In MCC and MCSEC, less than 10% of the student exercises were make-a-conjecture (MakeConj) exercises. In contrast, in CM, this category accounted for a little over 25% of the student exercises. CM recorded the highest number of MakeConj exercises. This was the only type of expected student activity, beyond investigate-a-conjecture exercises, evident in this textbook. Figure 2.15 shows a typical example of a make-a-conjecture exercise in CM.

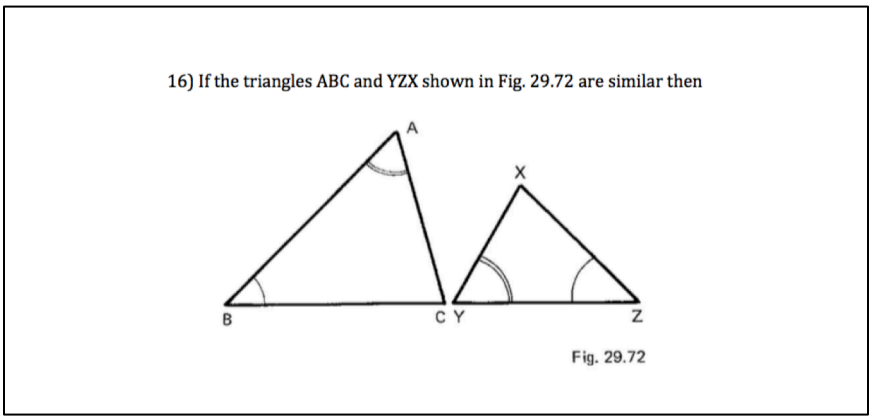


Figure 2.15: An example of a make-a-conjecture exercise. Adapted from "Certificate Mathematics" by A. Greer, and C.E. Layne, p. 222.

In this exercise students are given a geometric property of the two general triangles *ABC* and *YZX* as shown in the provided figure. Students are expected to make a conjecture using the given information. In this case the conjecture will take the form a logical implication, "If triangles *ABC* and *YZX* are similar then." This statement provides the hypothesis of the conjecture students can make. A possible solution for this exercise would expect students to state that if triangles *ABC* and *YZX* are similar then the ratios $\frac{BC}{AB}$ and $\frac{ZX}{YZ}$ are equal. Students can use the idea that in similar triangles the ratios of corresponding sides are equal. By identifying the corresponding sides and angles in the given triangles, students can use the given fact that these triangles are similar and make a suitable conjecture. The difference between this type of exercise and an investigate-a-conjecture exercise is the autonomy given to students to develop a conjecture based on the given information. In the InvestConj category, students are given a conjecture with the expectation of investigating its validity based on given information and knowledge about applicable geometric theorems and concepts.

Among the three textbooks, the fill in the blanks (FillBlank) type of expected student activity was either 0% or less than 10%. MCSEC was the only textbook, which offered FillBlank

exercises. The FillBlank exercises within the expected student activity related to mathematical claims required students to complete the missing components of a conjecture or mathematical claim, taken to be true. Additionally the FillBlank exercises also allowed students to complete an incomplete or partially completed proof argument. Figure 2.16 shows an example of a student exercise with an expected student activity related to mathematical claims within this category.

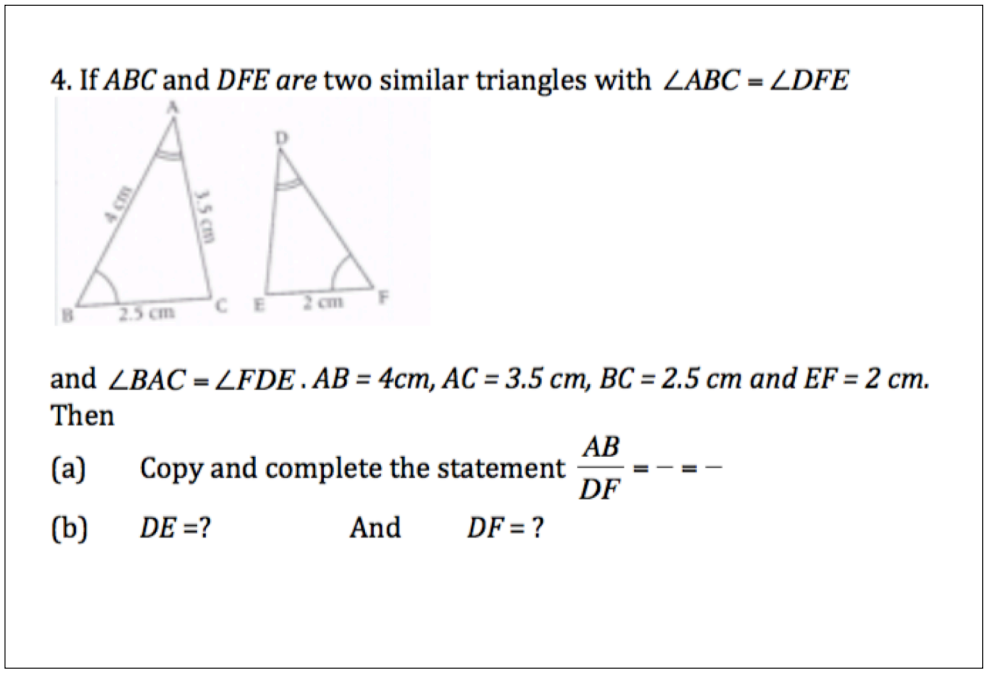


Figure 2.16: Fill in the blanks-type expected student activity. Adapted from "Mathematics for CSEC" by S. Chandler, E. Smith, F. W. Ali, C. E. Layne, and A. Mothersill, p. 147.

In this example students are given are given a mathematical claim about the similarity of the two given triangles *ABC* and *DFE*. In this exercise, students are given specific measurements about the sides and respective angles of the triangles. For example, the measures of angles *BAC* and *FDE* are equal. The mathematical claim in this exercise is written as a conditional statement using the words "if" and "then." The use of the conditional statement indicates that students are required to use the information given in the hypothesis, "If *ABC* and *DFE* are two similar triangles with $\angle ABC = \angle DFE$ and $\angle BAC = \angle FDE$, *AB = 4 cm, AC = 3.5 cm, BC = 2.5 cm and*

EF = 2 cm," to help complete the statement in the conclusion by identifying ratios equivalent to AB/DF. This statement in the conclusion contains blanks for students to fill using the given measurements and the fact that the triangles are similar. This example shows that within the expected student activity related to the mathematical claim, students are expected to fill in the missing components of the conclusion to the statement.

Geometric Calculation with Number and Explanation (CalcExp)

Of the different types of expected activities related to mathematical claims, I discovered Geometric Calculation with Number and Explanation (GCNE) or CalcExp-type[9] in two of the textbooks, MCSEC and MCC at 33% and 73% respectively. A GCNE exercise requires that students solve an algebraic-type problem for an unknown measure of a geometric figure. Students are required to use Geometry concepts, definitions, or theorems to help find the unknown quantity. Most importantly, the question explicitly requests that students provide explanations of their reasoning when finding the solution to the problem. In fact students are given an opportunity to develop a rationale for their algebraic calculations. Therefore I coded all GCNE exercises as developing a rationale with regard to the expected activity related to mathematical arguments.

For example, Figure 2.17 shows an exercise, which is coded as a CalcExp with respect to the expected student activity related to the mathematical claim and as developing a rationale for the student activity related to mathematical argument. In this exercise, students are expected to

[9] Throughout the results section I will refer to GCNE as CalcExp meaning to Calculate and Explain. I use the latter abbreviation for consistency with the naming trend I employed for the other categories of expected student activity. Additionally for brevity, I choose CalcExp to help with reading the results and easy recall of the characteristics of this type of expected student activity. However, it should be noted that I would maintain the GCNE acronym to give credit to the Geometric Calculation with Number (GCN) label in the literature from which I developed this type of student exercise, which is a unique characteristic of the textbooks in Trinidad and Tobago.

find the missing angles *DAB* and *ABC* and provide reasons for the steps taken in solving the problem. In my analysis, all CalcExp type exercises were considered as developing a rationale because they did not ask students to write a formal proof. The phrase "Give reasons for your statements," in Figure 2.17, indicates that students must provide mathematically acceptable reasons (i.e., logical arguments including theorems or definitions) to support claims they make about the angle measures.

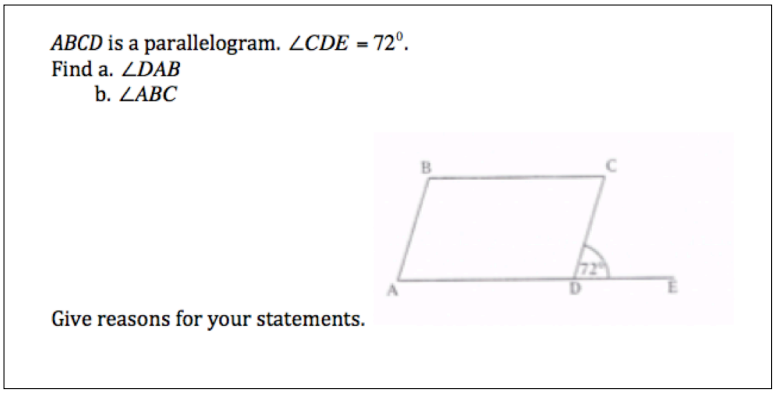


Figure 2.17: GCNE expected student activity. Adapted from "Mathematics for CSEC" by S. Chandler, E. Smith, F.W. Ali, C. E. Layne, and A. Mothersill, p. 143.

For example, as a solution to this exercise, students could label angle *DAB* as 72^0 with a supporting reason that because quadrilateral *ABCD* is a parallelogram, *AB* is parallel to *DC,* and the corresponding angles *CDE* and *BAD* formed by the transversal *AE* have equal measures. In this case, the students are not writing a formal proof for the result but are providing explanations, which illustrate their reasoning about the geometric properties of the given figure in the exercise. This example demonstrates how an opportunity for reasoning and proof is coded as an expected student activity related to mathematical claims and as related to mathematical arguments.

Overall, the CalcExp exercises require students to investigate mathematical claims by solving an algebraic problem using geometric theorems and definitions. More importantly, this type of exercise requested an explanation of students' reasoning to find the unknown

measurements in a given diagram. Although the latter requirement can possibly include inductive reasoning (i.e., measuring unknown values in a diagram), there exists the opportunity for students to use deductive reasoning to explain their computations. The use of deductive reasoning can possibly allow students an opportunity to explain how they use geometric theorems to solve the problem. In MCC, more than two-thirds (73%) of the expected activities were of this type. This textbook contained the highest percentage of this type of expected activity. In MSCEC, about one-third of the exercises were of this type (33%). CM did not contain any CalcExp exercises. With regard to expected activities related to mathematical claims, CM only offered opportunities for making and investigating conjectures.

Expected Student Activity Related to Mathematical Arguments

Table 2.13 shows the various expected activities related to mathematical arguments among the three textbooks. There was a significant statistical difference among the categories of activities within and between the textbooks. Within the expected student activities related to mathematical arguments, all three books offered exercises requiring the construction of a proof (ProofConst). CM had the largest percentage of proof construction activities, at 49% and MCC had the least at 26%.

In these exercises, students are expected to write a formal proof for a mathematical claim using logical arguments. A typical ProofConst opportunity would entail four properties. First, a student exercise framed within a Geometry context, whereby a mathematical claim is presented, which is yet to be proven. The mathematical claim would entail a hypothesis and a conclusion. The Geometry context in most cases would include a general or particular case of a geometrical object or situation.

Table 2.13

The Expected Student Activities Related To Mathematical Arguments

	No. of Expected Student Activity in Student Exercises Related to Mathematical Arguments			
Textbook	ProofConst (%)	RationaleDev (%)	FillBlank (%)	*Test Statistic*
CM	19 (49)	9 (23)	11 (28)	3.57*
MCC	48 (26)	137 (74)	0 (0)	156.72**
MCSEC	23 (43)	26 (48)	5 (9)	14.33**
Total	90	172	16	

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC; * = $p < .05$; ** = $p < .01$; *ProofConst = Construct a proof; RationaleDev = Develop a rationale; FillBlank = Fill in the blank*

Second, an accompanying diagram with all of the relevant geometric components available for inspection. Third, the author's use of the word "prove" which indicates that a logical deductive argument is expected which links the hypothesis to the conclusion, stated in the mathematical claim of the problem. Fourth, the author's expectation that the student has the responsibility for adding additional markings or auxiliary lines and writing a sequence of statements and supporting reasons linking the given hypotheses and the conclusion.

For example, Figure 2.18 presents a typical ProofConst exercise. In this exercise, students are given the geometrical measurements of two triangles formed by the intersection of two lines *AD* and *BC* as shown in Figure 2.18. Sides *AB* and *DC* are parallel and equal in length, which is indicated by the arrows and markings, respectively, in the given figure. The use of the keyword "prove" indicates that students are expected to construct proof arguments, which support the claim that the sides *AO* and *DO have equal length* as do the sides *BO* and *CO*. A typical proof will entail the use of geometrical theorems about the congruency of triangles *CDO* and *ABO*, to

construct supporting arguments for this claim.

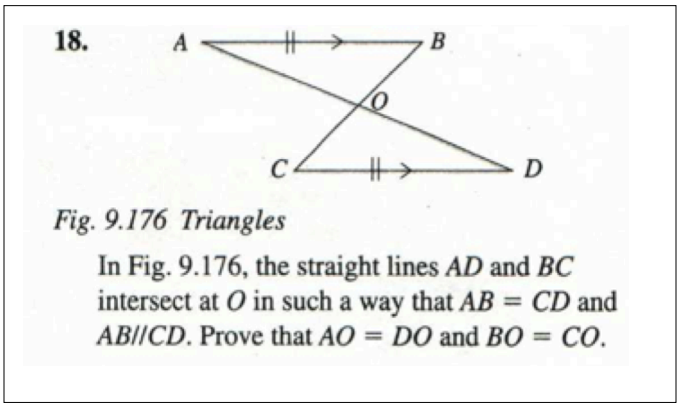


Figure 2.18: An example of constructing a proof exercise. Adapted from "Mathematics a Complete Course" by R. Toolsie, p. 457.

With regard to the expected student activities related to mathematical arguments, developing a rationale (RationaleDev) accounted for almost two thirds of these types of exercises (see Table 2.13). The activities, which offered the expected student activity of developing a rationale, were different than proof construction exercises because the author did not explicitly ask students to construct a proof. In all cases the author's stated, "Give reasons to support your answer" or "explain your reasoning," to convey their expectation of students. The omission of the word "prove" in these exercises suggested that the wording conveyed the author's intent of the possibility of students providing a non-proof argument or explanation to support their reasoning and claims. A typical RationaleDev exercise would expect students to provide an argument, which explains their reasoning when performing a geometrical computation. The solution keys for the exercises asserted this expectation of students. For example, a CalcExp exercise would be categorized as developing a rationale type of argument because students are expected to provide explanations to support the steps taken in performing a calculation. As I explained earlier I coded all CalcExp exercises as RationaleDev. As a result MCSEC had the 74% of its exercises coded as RationaleDev with respect to the expected activities related to

mathematical arguments. MCSEC and CM had 48% and 23% of its exercises within this category. CM, which did not have any CalcExp exercises, did offer opportunities that did not require students to construct a proof but expected students to provide a non-proof argument.

For example, Figure 2.19 shows an exercise taken form the section on quadrilaterals in CM. In this exercise, the authors ask students whether triangles *ABE* and *CDE* in the given figure are congruent. Students are also asked whether angles *DAC* and *DCA* are equal and if angle *DAB* is bisected by $\overline{AC}$.

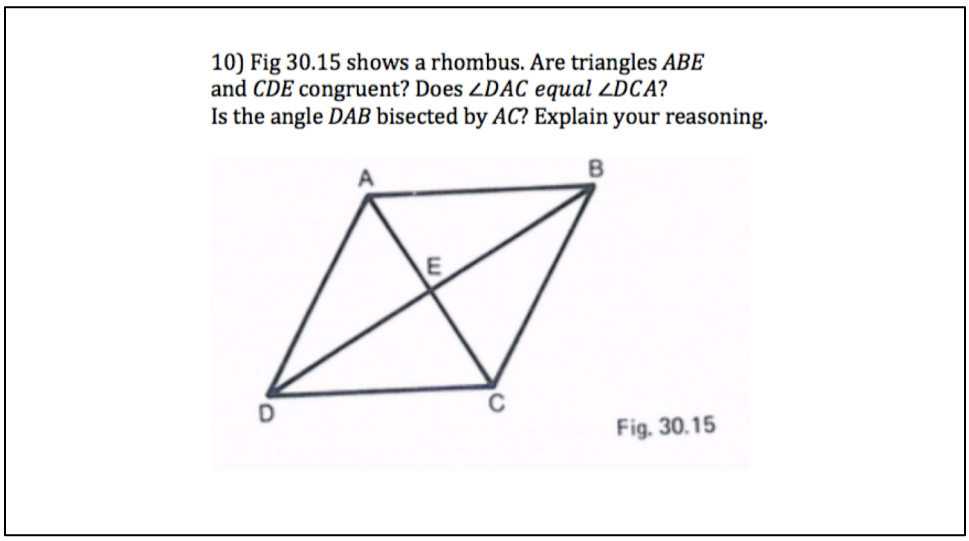


Figure 2.19: Example of a RationaleDev exercise. Adapted from "Certificate Mathematics" by A. Greer, and C.E. Layne, p. 227.

When the author used the phrase "explain your reasoning" it seemed to convey the author's intention for students to provide explanations that support their claims. For example, in answering the first question about the congruency of triangles *ABE* and *CDE*, a student is expected to use the idea that the given figure is a rhombus. Because all sides have equal lengths in a rhombus, *AB* is equal to *CD* and *DA* is equal to *BC*. Because opposite sides in a rhombus are parallel, $\overline{AB}$ and $\overline{CD}$ are parallel. Angles *ABE* and *CDE* are equal because they are the alternating

angles formed by the transversal $\overline{BD}$ cutting across these parallel lines $\overline{AB}$ and $\overline{CD}$. Additionally angles *AEB* and *CED* are equal because they are vertically opposite angels formed by the intersection of the diagonals $\overline{BD}$ and $\overline{AC}$. Therefore, because $\angle AEB \cong \angle CED, \angle ABE \cong \angle CDE,$ and $\overline{AB} \cong \overline{CD}$ it can be concluded that by the Angle-Angle-Side (AAS) congruency postulate, triangles *ABE* and *CDE* are congruent. Another possible solution would include and application of the Angle-Side-Angle (ASA) postulate.

A student must provide an argument equivalent to the previous explanation to support his or her claim. Due to the flexibility of the explanation requested by the authors, which induces a lower level of rigor than a formal proof argument, I considered this question as a RationaleDev rather than the ProofConst. Therefore both the wording and the practical expectation of the product of student's answers, I considered this question as a RationaleDev exercise.

Justification Types in the Student Exercises

In a manner similar to Hanna and de Bruyn (1999), Bieda (2010), and Otten et al. (2014), I used keywords such as "prove" to indicate exercises that would be justified deductively. This type of justification included proofs containing a logical chain of deductive arguments using geometrical theorems and definitions. Student exercises that required a deductive justification usually would have the keyword "prove" indicating that students use a formal proof argument as demonstrated in the worked examples to complete the exercise.

With regard to implicit justifications, I used the keywords such as "explain", "justify," or "show" and key phrases such as "explain why," "justify your reasoning," or "give reasons for your statements" to indicate that an implicit justification is required for a student exercise. I also considered the notion that students may possibly interpret explain and justify or show as requiring a proof argument. However, I followed Otten and colleagues definition of the

justification types because my focus was on the opportunities for reasoning and proof as stated in the textbooks rather than interpreting students reasoning. My reason for this was firstly to follow Otten and colleagues approach to coding the justification types. Secondly, similar to Otten and colleagues I wanted to distinguish the adaptation of their framework as one focusing on the opportunities for reasoning and proof rather than students' reasoning. Therefore, to distinguish these exercises I coded those in which the authors requested students to explain why an argument as requiring a deductive justification (i.e., a proof). I coded exercises as implicit justifications in cases where the authors asked students to explain their reasoning or why they took certain steps in a calculation as implicit justifications.

In Table 2.14, I summarize the distribution of the various types of justifications in the student exercises of the textbooks. Within the textbooks, there were no statistically significant differences between the types of justification in the student exercises of CM. However in MCC and MCSEC, there was a statistical significant difference between the types of justifications in their respective student exercises.

Table 2.14

Types of Justifications in the Student Exercises

	No. of Justification types (%)			
Textbook	Deductive (%)	Implicit (%)	*Empirical*	*Test Statistic*
CM	19 (49)	20 (51)	0 (0)	-0.16
MCC	50 (27)	135 (73)	0 (0)	-6.25**
MCSEC	22 (41)	32 (59)	0 (0)	-1.36*
TOTAL (%)	91 (33)	187 (67)	0 (0)	

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC; * = $p < .05$; ** = $p < .01$

In all three textbooks the implicit justifications were the most frequently occurring exercise type. Implicit justification exercises accounted for 51% to 73% of the exercises in the textbooks. In MCC, about three-quarters of the exercises required implicit justifications. This means that the student exercise requested that students engage in reasoning and proof (e.g., "Explain with reasons why" or "Give reasons for your statements") but did not explicitly specify the nature of the argument to be produced. The open-endedness of the type of argument expected in the aforementioned phrases indicates that the exercise is giving students the agency to determine why the argument is important for understanding the mathematical claim. Thus, in my analysis, all CalcExp (i.e., GCNE) exercises were considered as exercises that would be justified implicitly. In MSCEC and CM, implicit justifications were expected for 59% and 51% of the exercises respectively. The remaining student exercises in each text expected deductive justifications.

Overall, MCC had the largest imbalance between the types of justifications expected in the student exercises, with 73% implicit justification exercises compared to 27% deductive justification exercises. As stated earlier, my analysis suggested that the justification type of all GCNE exercises were implicit. Therefore, the higher proportion in the implicit justification in MCC reflects the higher proportion of GCNE exercises in this textbook. However, CM had a more balanced distribution of the justification types in the student exercises. In CM, roughly half of the exercises required deductive justification and the other half required implicit justifications. Overall, all three textbooks had over 50% of their justifications being implicit, with the deductive justifications ranging from 27% to 49%. None of the exercises in any of the books required empirical justifications.

Statements about the Practice of Reasoning and Proof

To answer my first research question: *What is the nature and distribution of the opportunities for reasoning and proof within the Geometry sections of secondary school textbooks in Trinidad and Tobago?* I describe in this section the nature of the statements about the practice of reasoning and proof in the textbooks. In my analysis, I focused on the statements as presented in the textbook expositions to ascertain how the authors afforded students opportunities to engage in or reflect on reasoning and proof. I considered the categories aligned with the various activities associated with reasoning and proof as defined in Stylianides (2009).

Table 2. 15

Classification of Statements about Reasoning and Proof in Student Exercises

Textbook	No. of Statements about Reasoning and Proof				
	Identifying Patterns (%)	Making Conjectures (%)	Providing Non-Proof Arguments (%)	Constructing a Proof (%)	*Test Statistic*
CM	2 (17)	0	0	10 (83)	22.66**
MCC	16 (39)	12 (29)	7 (17)	6 (15)	6.32**
MCSEC	1 (3)	1 (3)	8 (27)	20 (67)	27.25**
Total	19 (23)	13 (16)	15(18)	36 (43)	

** Indicates a statistical significant difference between the types of opportunities about the practice of reasoning and proof.*

Table 2.15 shows the number and percentage of statements about reasoning and proof in the textbook expositions. As described earlier these statements consists of narrative text which talk about or motivate students to think about (a) identifying patterns, (b) making conjectures, (c) providing non-proof arguments, or (d) constructing proofs. There was a statistically significant difference among these categories within each textbook. Among the three textbooks,

MCC had the highest number of such statements, (49%) of all statement about the practice of reasoning and proof. Whereas, CM having the lowest approximately 14%.

In MCC, the book with the highest number of statements about the practice of reasoning and proof, approximately a little over two-thirds of these statements were about identifying patterns (39%). All of these statements were found in the class activities in the various expository sections in this textbook. For example, in Figure 2.20, I have shown an activity taken from the section on quadrilaterals and polygons in MCC. In this activity, the authors asked students to identify the patterns for the sum of the interior angles of various polygons. The given table with the missing items provided students with the opportunity to obtain a generalized pattern for the formula shown in the last row of the table. This activity demonstrates the author's apparent intent of possibly allowing students to think about the patterns in the formulas for the sum of interior angles for the polygons. A possible approach to this exercise could allow students to use the patterns in the earlier rows to make a generalized formula for any *n-gon*. Another option could include students identifying the pattern in the final row for an n-gon and apply it to the other rows. The main tenant of this exercise is the author's intention for students to reason about the patterns they identify in the table to complete the task.

THEOREM 1: *The sum of the interior angles on an n-sided polygon is (2n-4) right angles or 90^0 (2n-4) or 180^0(n-2).*

CLASS ACTIVITY

Draw polygons with the stated number of sides in Table 9.6. Then choose one vertex and from it draw diagonals until no more can be drawn. Hence fill in the missing information in Table 9.6 and confirm the given information.

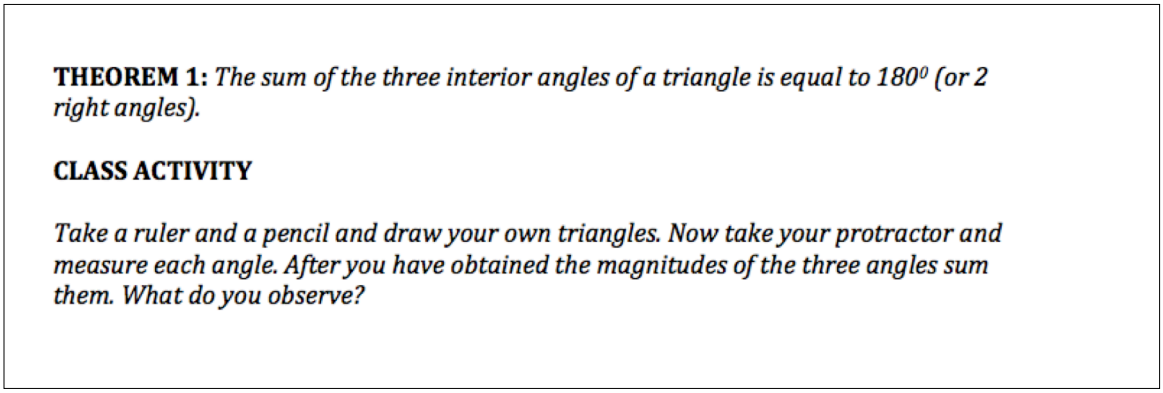
Table 9.6

Name of polygon	*Number of vertices*	*Number of sides*	*Number of triangles obtained*	*Sum of interior angles*
Triangle	*3*	*3*	*1*	$1\times180^0=180^0$
Quadrilateral				$2\times180^0=360^0$
Pentagon				$\times180^0=540^0$
Hexagon				$\times180^0=720^0$
Heptagon				$\times180^0=900^0$
Octagon				$\times180^0=1080^0$
Nonagon				
Decagon				
Undecagon				
Dodecagon				
Icosagon				
n-gon	*n*	*n*	$n-2$	$\times180^0=$

Figure 2.20: Example of an intended class activity for identifying patterns. Adapted from "Mathematics a Complete Course" by R. Toolsie, p. 479.

In addition to identifying patterns, some of the class activities also seemed to offer opportunities for student to make conjectures about the patterns they would identify. For example, Figure 2.21 shows an activity asking students to explore the theorem about the sum of the interior angles of a triangle is 180 degrees. In this activity, students were expected to construct various triangles, measure the interior angles of each triangle, and then find the sum of the interior angles. When the author stated, "What do you observe?" it seemed to suggest that students are prompted to identify a pattern concerning the sum of the interior angles of the various triangles they have drawn. The phrase also suggests that students are expected to make a statement about their observations of a few cases and explain their reasoning about any claims

made about the patterns they observed. Because the students are expected to make claims based on a few cases, there exists some measure of uncertainty about the validity of these claims. As a result, students' activity here would be considered as also making a conjecture.



THEOREM 1: *The sum of the three interior angles of a triangle is equal to 180^0 (or 2 right angles).*

CLASS ACTIVITY

Take a ruler and a pencil and draw your own triangles. Now take your protractor and measure each angle. After you have obtained the magnitudes of the three angles sum them. What do you observe?

Figure 2.21: An example of an intended class activity for reasoning and proof. Adapted from "Mathematics a Complete Course" by R. Toolsie, p. 443.

In MCC, the author explained in the preface of the book that the suggested class activities are provided as an opportunity for students to investigate mathematical properties. The author also suggested that these activities could be done before introducing the formula, theorem or mathematical concept being investigated in the activity. Therefore, in the cases where a teacher may choose to use the activity before presenting the interior angles sum theorem, the author possibly intended for students to engage in the practice of moving from pattern identification to making a conjecture about the sum of the interior angles of a triangle. Therefore the teachers' use of the textbook may influence the author's intention for students to engage in or reflect on the activities of pattern identification and conjecturing.

In MCC, less than 20% of the statements were intended to elicit non-proof arguments or constructing proofs. In contrast, MCSEC included 67% of the statements that were intended to elicit proofs, with 27% about providing non-proof arguments to support mathematical claims. Approximately 3% of the statements in this textbook offered opportunities for students to think

about or engage in pattern identification or making conjectures (see Table 2.15). In CM, the majority of the expository statements were about constructing proofs (83%), with the remaining 17% of statements about identifying patterns. There were no statements in this textbook's expositions motivating students to reflect on the development of non-proof arguments or making conjectures. Overall, a substantial percentage (43%) of all statements about reasoning and proof either engaged students in or motivated students to reflect on the practice of constructing proofs. Approximately less than 20% of the statements were about either making conjectures or providing non-proof arguments. With 23% of the statements offering opportunities for students to engage in or reflect on identifying patterns.

Distribution of Opportunities about the Practice of Reasoning and Proof

To answer my first research question, which asked: *What is the nature and distribution of the opportunities for reasoning and proof within the Geometry sections of secondary school textbooks in Trinidad and Tobago?* I also present here the distribution of the statements about the practice of reasoning and proof. These include statements, which motivate students to engage in or reflect on pattern identification, making conjectures, developing non-proof arguments, and constructing a proof. For example, Figure 2.21 presents an example of an opportunity about the practice of reasoning and proof in MCC in the topics Triangles. Table 2.15 presents the distribution of the statements according to the type among the three textbooks. Table 2.16 presents the distribution of the statements according to the topics selected for my analysis. There was a statistical significant difference for distribution of the number of reasoning and proof statements about reasoning and proof among the Geometry topics in CM and MCC. However there was no statistically significant difference in MCSEC.

Table 2.16

Distribution of Statements about Reasoning and Proof among Topics in Geometry

Textbook	No. of statements about the practice of reasoning and proofs						
	Triangles	Congruent Triangles	Similar Triangles	Pythagoras' Theorem	Quadri-laterals	Circles	*Test Statistic*
CM	1 (8)	5 (42)	5 (42)	1 (8)	0 (0)	0 (0)	25.33*
MCC	9 (24)	5 (14)	3 (8)	0 (0)	5 (14)	15 (40)	35.4**
MCSEC	3 (10)	6 (20)	3 (10)	1 (3)	8 (27)	9 (30)	20.11*

Note. CM = Certificate Mathematics; MCC = Mathematics a Complete Course; MCSEC = Mathematics for CSEC; * = $p < .05$; ** = $p < .01$

In MCC, the topics Circles and Triangles offered the most statements about the practice of reasoning and proof, 40% and 24% respectively, with the topic Similar Triangles offering the least (8%). In MCSEC, the sections on Circle Geometry and Quadrilaterals had the most statements about the practice of reasoning and proof at 30% and 27% respectively. However, the sequence of topic was different in this textbook (see Table 2.5 in the Methods section). The ordering of the topics in MCSEC was not in the same as the other two textbooks. In MCSEC, the authors opted to place the topic Quadrilaterals after Triangles before introducing congruency and similarity. As I explained earlier, the authors' sequencing of these topics is important because it determines the prior knowledge and necessary resources students should possess for successfully completing exercises. Therefore the proof of a mathematical claim may be different according to what students are given to in previous sections.

In MCSEC the percentage of statements the authors presented about the practice of reasoning and proof increased from 10% in Triangles to 27% in the section on quadrilaterals. The placement of this topic by the authors may have afforded students more opportunities in this

topic to engage in or reflect on the practice of reasoning and proof before introducing them to Congruent Triangles. However in CM and MCC the percentage of opportunities in the topic Quadrilaterals was not as substantial as compared to MCSEC.

In CM, the topics Circles and Quadrilaterals did not offer any statements about reasoning and proof. Similar Triangles and Congruent Triangles both offered the most statements about reasoning and proof (42%) as shown in Figure 2.22. This finding is in contrast to the other findings of the analyses of the other textbooks, in which Circles and Quadrilaterals offered a substantial proportion of statements about reasoning and proof.

Summary

Figure 2.22 shows the distribution of the statements about the practice of reasoning and proof in the six topics among the three textbooks. Overall, in MCC and MSCEC, the topic Circles had the highest occurrence of statements about the practice of reasoning and proof. However, in CM, there were no statements about the practice of reasoning and proof in this topic. The topics, Congruent Triangles and Similar Triangles contained the highest occurrence of statements about reasoning and proof in this textbook, at 42%.

Among all three textbooks, MCSEC was the only textbook that offered opportunities for students to engage in reasoning and proof in all six topics (see Figure 2.22). MCC and CM offered opportunities in five and four out of the six topics respectively. All three textbooks offered reasoning and proof opportunities in the topics: Triangles, Congruent Triangles, and Similar Triangles. At least two of the textbooks offered opportunities about reasoning and proof in Pythagoras' Theorem, Quadrilaterals, and Circles. Although all three books offered opportunities about the practice of reasoning and proof, there was an uneven distribution among the Geometry topics.

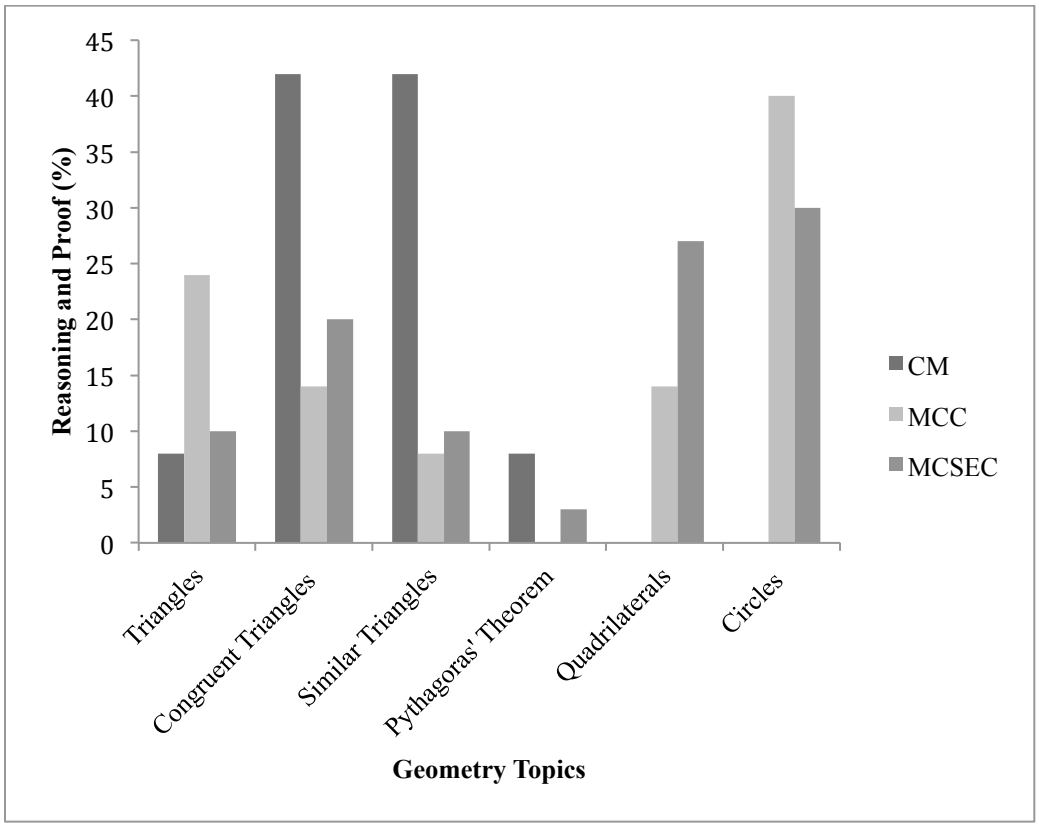


Figure 2.22: Distribution of Statements about Reasoning and Proof

For example, CM did not offer opportunities in two of the topics, however over 40% of the opportunities were distributed in each of the topics Congruent Triangles and Similar Triangles. Additionally, even though MCSEC had opportunities in each topic, the opportunities ranged from as high as 30% in Circles to as little as 3% in Pythagoras theorem.

Comparing Textbook Expositions to Student Exercises

To answer my second research question (*How do the reasoning and proof opportunities in Geometry sections of the textbook expositions compare to the opportunities in student exercises?),* I discuss the general nature of the mathematical statements followed by the justifications. Figure 2.23 compares textbook expositions and student exercises with respect to the percentage of opportunities for reasoning and proof that were general in nature for each portion of the textbook. There was a statistical significant difference *($p = 0.000211$)* between the

mathematical statements type in the expositions and student exercises of the three textbooks. In CM, 77% of the reasoning and proof opportunities in the exposition involved general statements, whereas 26% of the reasoning and proof exercises involved such statements. Figure 2.23 also shows the contrast between the types of mathematical statements represented in the textbook exposition and those in the student exercises. For example, my analysis suggests that at least three-quarters of statements in the textbook expositions (75%), and as many as 87% were general in nature. In contrast, less than 30% and even as little as 3% of the statements within the reasoning and proof exercises were general.

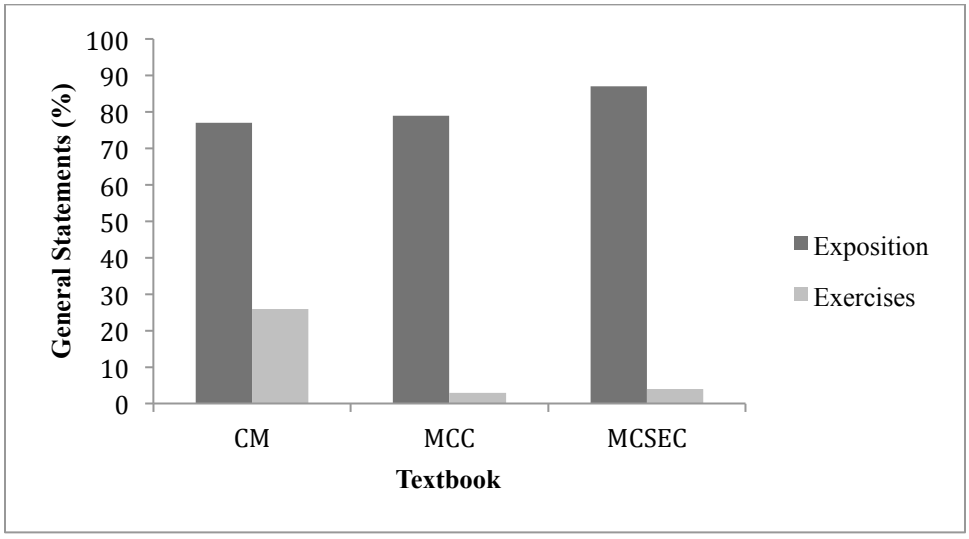


Figure 2.23: General mathematical statements in textbook exposition versus student exercises

Figure 2.24 also demonstrates the contrast between the particular mathematical statement types in the textbook expositions and exercises. For example, in MCC, 85% of the statements in the exercises were particular in nature, whereas 21 % of the statements in the expositions were particular. Furthermore, in cases where particular statements did appear in expository sections they were usually in the form of worked examples. Conversely, when general statements appeared in student exercises, they often were accompanied by a particular instantiation of the general claim.

Of the mathematical statements presented in the textbook expositions, many were justified deductively but many were also justified empirically. This means that both proofs and examples were utilized to justify statements or worked examples in the textbook expositions. As shown in Table 2.6, all worked examples were justified deductively. In MCSEC, the class activities in the textbook expositions mainly used empirical arguments to justify claims or conjectures made by the authors or expected by the students.

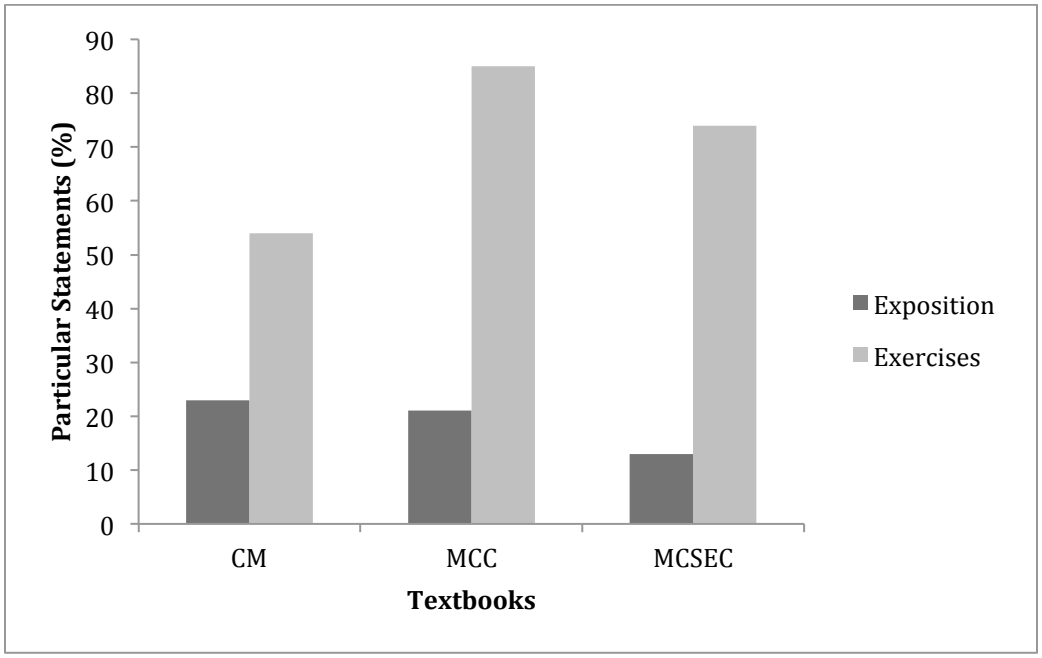


Figure 2.24: *Particular mathematical statements in textbook exposition versus student exercises*

Within the exercises, the types of justification to be provided by the students were most often implicit. In my analysis and in agreement with, Otten and colleagues (2014) I considered that students interpret "prove" as requiring a deductive argument. Therefore I coded the justification of exercises with the keyword "prove" as deductive. However, I coded exercises explicitly requiring an explanation or the provision of reasons as implicit justifications. My reason for this distinction is that according to Otten and colleagues, students may not necessarily interpret prompts to explain their reasoning as opportunities to construct a deductive argument.

As a result of the classification of students' interpretation of the keywords explain and prove, my analysis suggest that implicit justification were more prevalent among the reasoning and proof exercises. In the reasoning and proof exercises, approximately two-thirds contained prompts for students to explain their reasoning. Therefore, approximately two-thirds of the exercises required implicit justifications.

Finally, concerning the statements about the practice of reasoning and proof, the textbook expositions contained more statements engaging students in the construction of proofs (46%). Only 15% of these statements referred to the practice of developing non-proof arguments. However in the student exercises there was a reversal of this result. Approximately only 33% of the reasoning and proof exercises offered opportunities for students to construct proof meaning the exercises expected deductive justifications. Whereas, 67% of the reasoning and proof exercises expected students to develop non-proof arguments meaning they required implicit justifications. Overall, in the textbook expositions, of the mathematical statements about reasoning and proof, 21% were not justified. Of the 519 exercises in the selected Geometry topics, 54% offered opportunities for reasoning and proof.

Overall these findings in the previous sections indicate that the three textbooks in this study had at least one third of their student exercises offering opportunities for reasoning and proof. In CM, the majority of opportunities afforded students' engagement or reflection on the construction of proof arguments. Additionally this textbook afforded opportunities for students to make conjectures. However there were limited student exercises offering pattern identification or non-proof arguments. MCC had the lowest percentage of exercises offering the construction of proof. Nonetheless there was a substantially number of opportunities for developing rationales due the prevalence of GCNE–type exercises. This textbook also contained exploratory activities,

which afforded opportunities for pattern identification and conjecturing. These unique types of activities provided the trajectory of reasoning and proof practices from initial pattern identification to making conjectures about observed patterns. In MCSEC, students were provided opportunities to engage in all the practices of reasoning and proof. Furthermore, among the three textbooks, the authors of MCSEC used the widest variation of justification types in their demonstrations of proof arguments in the textbook expositions. These included outlines of proof arguments, left to students as an exercise, and fill in the blanks-type of justifications. All three textbooks offered reasoning and proof opportunities in the sections on Congruent and Similar triangles. However, in MCC and MCSEC, a substantially proportion of opportunities were distributed in the sections on Quadrilaterals and Circles. In summation, the textbooks offered opportunities for students to engage in and reflect of the process of reasoning and proof.

Discussion

In this section, I present a characterization of each textbook according to the opportunities for reasoning and proof. For each textbook, I discuss the unique features that enable students to engage in or reflect on the practice of reasoning and proof. I also discuss the overall implications of my findings as it relates to the recommendations in policy documents, teaching, and students learning of reasoning and proof in Trinidad and Tobago. I also introduce the characterization of the textbook according to the four competing modal arguments for the Geometry curriculum in secondary school mathematics (González & Herbst, 2006). The latter characteristic is useful for understanding the role of Geometry in the promotion of opportunities for reasoning and proof in students' secondary school mathematics experiences in Trinidad and Tobago. This is important because it would help researchers determine whether the textbook authors' promotion of reasoning and proof in Geometry allow students to see the intellectual

need of these activities. Additionally, researchers can determine whether the promotion of reasoning and proof aligns with the reformers' advocacy for these activities in the learning of Geometry in Trinidad and Tobago. In the following sections, I present the characterizations for each textbook based on my analysis according to the theoretical framework. I also summarize the three characteristics in Table 2.17.

Certificate Mathematics

Certificate Mathematics (CM) offered the most opportunities for constructing proof arguments. For example, of the 110 student exercises in the Geometry sections in CM, 35% offered opportunities for reasoning and proof. In CM, about one half of these opportunities for reasoning and proof, expected students to construct a proof argument. This finding is important because the construction of proof arguments allows students to use formally introduced theorems and concepts to construct logical deductive arguments that explain why a result may be true (Stylianides, 2009). Proof construction also affords students with opportunities to use their mathematical knowledge to practice deductive reasoning. By deductive reasoning, I refer to students' use of geometrical properties in a given diagram, or known geometrical theorems and concepts to formulate arguments that support claims they make when proving a result. The formulation of proof arguments is considered as the most important and culminating activity in the reasoning and proof process (Hanna, 2000; Stylianides, 2009; Otten et al., 2014).

CM, unlike the other two textbooks MCC and MCSEC, does not offer opportunities for students to engage in pattern identification, or writing non-proof arguments. However, CM did promote opportunities to make conjectures. Nonetheless, the absence of pattern identification and creating non-proof arguments in this textbook could indicate to students that these two activities do not play an integral role in reasoning and proving. Therefore potentially limiting the

entry points for students to engage in reasoning and proof. The prevalence of the construction of formal proof arguments does indicate the authors' emphasis on proof being a formative opportunity for students to develop logical thinking skills immediately without the trajectory of reasoning and proof processes from pattern identification.

In addition to constructing proof arguments, CM did provide students with opportunities to develop conjectures. More importantly, CM offered opportunities to investigate the validity of conjectures. However, these two activities were distinct in that students were expected to only investigate given conjectures. Approximately 74% of the reasoning and proof opportunities promoting conjecturing required that student investigate the validity of conjectures provided by the authors. Students were not afforded opportunities to further test the conjectures they created. Therefore this may promote the belief in students that making a conjecture is an end point in the proving process as opposed to merely a precursor for developing a proof. Several researchers identified the testing of conjectures as an integral step in the work of mathematicians in their quest to prove mathematical claims made after identifying patterns and making conjectures (e.g., Lampert, 1990; Pólya, 1968; Schoenfeld, 1992; Stylianides, 2009). Therefore it would be ideal that CM included tasks that specifically required that students test the validity of conjectures after they develop them.

Table 2.17

Summary of Characteristics of Secondary School Textbooks in Trinidad and Tobago

Certificate Mathematics (CM)	Proof Construction	Emphasis on the construction of proof. Aligns with the policy documents in Trinidad and Tobago, which claim "students must be given opportunities to develop logical deductive arguments" (Republic of Trinidad and Tobago, Ministry of Education, 2009, § 2: 1). CM encapsulates the vision of the reformers emphasis of the need for students to write proof arguments.	Supports the formal argument that the goals of the Geometry content in the textbook are the development of logical reasoning skills through proof construction.
Mathematics a Complete Course (MCC)	Developing non-proof arguments GCNE- type exercises Class activities promote pattern identification and conjecturing	Exemplify the promotion of explanatory proofs therefore fulfilling the objective of making connections among mathematical properties (Republic of Trinidad and Tobago, 2009).	Supports the mathematical argument by engaging students in activities reflective of the work of mathematicians.
Mathematics for CSEC (MCSEC)	Provides opportunities for all activities of reasoning and proof (i.e., pattern identification, conjecturing, developing non-proof arguments, and proof construction).	Secondary school students should be given opportunities to engage in pattern identification, conjecturing, formulating proof arguments and non-proof arguments throughout their mathematical experiences (Trinidad and Tobago, 2005, 2009).	Promotes the utilitarian argument by allowing students to study geometric concepts through job-oriented applications or modeling real-world situations.

In addition to construction of proof arguments, with regard to the expected activity for mathematical arguments, CM did offer fill-in-the-blank (FillBlank) type of proof arguments. FillBlank proofs are partially completed proofs. In CM, the authors presented a statement in a proof argument with missing components for students to insert so that the statement is logical, meaning that is adheres to one of the mathematical logic argument structures such as "If statement, then statement" or hypothetical syllogism[10]. Furthermore the missing components must also ensure that the statement can be deduced form the preceding statements in the proof arguments. McCrone and Martin (2004) reported that students performed better on the fill in the blank proof tasks than on complete proofs. Although their claim is based on the U.S. type of fill-in-the-blank proofs in which students are usually given two column format proofs with missing information, the support mechanisms provided by the FillBlank type of arguments in CM could possibly reduce the complexity of constructing a proof argument.

Another noteworthy feature about the proof arguments presented in CM is the structure. The demonstrations of proof arguments presented in the textbook expositions in CM and the other textbooks were typically in a paragraph type style. Paragraph style proofs utilize everyday language and written sentences to convey proof arguments. A complete sentence is equivalent to one of the supporting reasons that link the hypotheses to the conclusion. Some researchers support this format because it allows students to develop arguments without the rigidity of a prescribed format to support the validity of conjectures they make in the reasoning and proof process (Cirillo, 2008; Cirillo & Herbst, 2012; Hanna & de Villiers, 2008). These researchers noted that a paragraph style proof provides the opportunity to construct proof arguments with the

[10] Hypothetical syllogism is a logical implication linking three statements, whereby the premises are if statement 1 implies statement 2, and statement 2 implies statement 3, then we conclude that statement 1 implies statement 3.

use of linguistic tools that frame the mathematical deductions in the arguments they wish to convey. Students can use language readily available to explain their reasoning in a conversational style. The prevalence of this format in the textbooks suggests the expectation in this textbook that proofs do not need to adhere to the two-column format used in the United States. CM's demonstrations of proof arguments suggest that proofs can be written in paragraph style, ensuring the argument is logical and each statement or sentence is deduced from the hypothesis to the conclusion. Paragraph proofs can be less intimidating to students because they are like ordinary writing and are not confined by the requirements of a structured mathematical format (Cirillo & Herbst, 2012). However this format has received criticism by teachers because students who use it often exclude their reasoning and make incorrect deductions (Cirillo, 2008). Thus, the opinion that teachers may have about this type of proof representations can impact the way they use the representations of proof in the textbooks during instruction. Nonetheless, despite the style of proof promoted in textbooks, the embedded opportunities for students to engage in constructing proofs help them experience the work done by real mathematicians.

Several researchers claimed that a common idea in the work of a mathematician is the exploration of mathematical relationships to identify and arrange significant facts into meaningful patterns. By using these patterns, mathematicians formulate new conjectures, testing these conjectures against new evidence, and revision of the conjectures that conform to the evidence. Finally providing informal arguments that demonstrate the validity of the conjectures (e.g., Hanna, 2000; Lakatos, 1976; Pólya, 1954; Stylianides, 2009). The culmination of these activities is the construction of proof. These aforementioned activities help mathematicians in understanding mathematical relationships, building a foundation or the development of proof (Stylianides, 2009). Therefore the inclusion of proof with the preceding activities help students

sees the necessity of proof in a mathematician's work and the need for these activities in developing proofs. Furthermore a textbook's presentation of proof in conjunction with these activities that support its development can afford students an important kind of scaffolding. Pólya (1968) and Stylianides, (2009) noted that this is the type of scaffolding that more advanced users of mathematics have available as they establish and make sense of mathematical knowledge.

Stylianides (2005) noted that mathematicians themselves might not be involved in all stages of the proof development process: identification of patterns, conjecturing, developing non-proof and constructing proof arguments. While this may be true, it is important that textbooks integrate the trajectory of these activities of reasoning and proof in the opportunities provided to students (Davis 2012). The importance of this trajectory is that it allows students to see the need of transitioning from inductive to deductive reasoning when developing their mathematical knowledge (Otten et al., 2014; Stylianides, 2009). For example, through pattern identification and conjecturing students have the opportunity to reason around a few cases and see a common characteristic which can further lead to their hypothesizing or conjecturing. When making conjectures there is some measure of uncertainty (Stylianides, 2009), thus students must test the conjectures they develop. As students test conjectures they use their mathematical knowledge to deduce arguments that explain why their conjecture is valid. Therefore students engage in deductive reasoning. Additionally, students in deducing arguments may see limitations in their conjectures, meaning they establish whether the generality of their conjecture holds. Through these activities students can eventually write a logical deductive argument linking a conclusion from hypotheses. The process of proof construction through pattern identification and conjecturing is a formal way of expressing certain kinds reasoning and justification (NCTM,

2000). These kinds of reasoning include both inductive and deductive reasoning.

Overall CM placed an emphasis on the construction of proof. This is in accordance to the policy a document in Trinidad and Tobago, which claim "students must be given opportunities to develop logical deductive arguments" (Republic of Trinidad and Tobago, Ministry of Education, 2009, § 2: 1). The vision of the reformers emphasizes the need for students to write proof arguments. Additionally, the CSEC mathematics syllabus highlights the importance of writing proofs in students' mathematical experiences. In their annual subject reports based on students performance on proof items in the CSEC examinations, the examiners suggested that: "Teachers should provide opportunities for students to develop logical reasoning through proof writing" (CXC Subject Award Committee, 2014, p. 23).

Furthermore the CM textbooks promote views of the competing *formal argument* for Geometry in the high school mathematics curriculum presented by González and Herbst (2006). In their analysis of historical documents, textbooks, and U.S. national standards, González and Herbst presented several competing arguments for the high school Geometry course in the United States. Although these arguments are based on the U.S. non-integrated mathematics curriculum and a single Geometry course, they provide insights about the varying role of proof and Geometry in students' mathematical experiences. In the case of Trinidad and Tobago, there exists an integrated curriculum, which incorporates the Geometry content with other mathematics topics such as Algebra, Calculus, and Number Theory throughout the secondary school curriculum. The manner in which the authors of CM promote opportunities for students to engage in or reflect on constructing proof arguments supports the formal argument that the goals of the Geometry content in the textbook is that Geometry fosters the development of logical reasoning skills. The authors of CM promote the Geometry content as a topic in the curriculum

in which students develop their ability to write logically deduced arguments, and these arguments validate or explain why a mathematical claim can be true. The limited opportunities for students to engage in the preliminary reasoning and proof processes of finding patterns coupled with the emphasis on proof indicate the value of proof promoted by the authors of CM. This suggests that the authors of the CM textbook highly value proof writing as a fundamental opportunity for students' metacognitive development of logical reasoning skills. Therefore Geometry is presented in this text as an ideal opportunity to develop such skills.

Mathematics a Complete Course

Mathematics a Complete Course (MCC) contained the highest number of exercises of all three textbooks. Additionally this textbook also offered the highest percentage of reasoning and proof opportunities (77%). MCC contained exercises, which promoted pattern identification, conjecturing, and developing non-proof argument. However, MCC offered the lowest percentage of proof construction opportunities among the three textbooks. Although the sections on Congruency and Similarity of Triangles had approximately 90% of the exercises offering opportunities for constructing proofs, the overall percentage of opportunities for all the Geometry sections was low at 26% compared to the other textbooks. Nonetheless, the textbook did provide the type of scaffolding through pattern identification and conjecturing that could facilitate the construction of proof arguments.

The author's promotion of pattern identification and conjecturing was highlighted in the inclusion of class activities. As described earlier in the results section, this unique feature of the textbook afforded students the opportunity, through empirical investigations, to discover geometrical patterns in several objects. For example, students were asked to construct several triangles or quadrilaterals and observe a geometrical pattern such as the sum of interior angles.

These initial explorations are integral in initiating the mathematical activities that eventually culminate in the construction of proofs (de Villiers, 2010; Lakatos, 1976; Pólya, 1954; Sandefur, Mason, Stylianides, & Watson, 2013; Stylianides, 2009). These researchers agree that constructing examples, as advocated by Pólya and Lakatos is not usually to accumulate empirical data but rather to see the general through the particular. The particular case should be treated as a means of seeing important patterns that can be generalized to develop a conjecture that can be further investigated for its validity and generality.

Additionally in some activities students were asked, "What do you observe?" The use of this phrase encouraged students to make conjectures about the patterns they observed. In some cases students were given the opportunity to investigate the validity of the conjecture they made and then provide supporting reasons why it's validity holds. These class activities provide the type of initial scaffolding needed for students to eventually develop proof arguments. Harel and Sowder (1998) noted that a crucial point in a mathematician's work is spent on exploring and conjecturing. This step helps in generalizing one's claims and arguments and leads to constructing a proof that holds for all cases. Therefore the inclusion of class activities in MCC, which promote conjecturing from initial explorations, support the type of school mathematical experiences that reflect the type of reasoning and proof done by mathematicians.

Another unique feature of MCC was the substantial offering of what I defined as Geometric Calculation with Number and Explanation (GCNE) at 73% of the student exercises. In the results section I also referred to these opportunities as CalcExp, meaning to calculate and then explain the steps for computations. This type of reasoning and proof opportunity afforded students to develop supporting non-proof arguments. Several researchers have described a similar type of exercise called Geometric Calculation with Number (GCN) (Ayres & Sweller,

1990; Küchemann & Hoyles, 2002; Hsu & Silver, 2014). As I explained earlier, these two types of exercises are similar in their expectation for students to use geometric theorems and definitions to construct supporting reasons for the steps taken when performing a geometric calculation. This calculation involves the use of algebraic techniques to find a missing component of a geometric figure such as an angle or side. GCN and GCNE both require an explanation for steps taken in a calculation, however their difference lies in how the exercise implicitly or explicitly requests the explanation.

In the case of the GCNE exercises, the authors explicitly would use the phrases " Give reasons for your statements" or "Explain with reasons why" to indicate that supporting explanations are required for full credit for the exercise. The explicit requirement suggests the value the authors place on student's explaining their reasoning. Although GCN exercises do not explicitly requests explanations as GCNE, they both promote opportunities for students to see the need for the explanatory role of proof in their mathematical experiences. Therefore the GCNE exercises in MCC textbook exemplify the promotion of explanatory proofs therefore fulfilling the objective of making connections among mathematical properties. This is outlined as a learning outcome of the one of six standards in the current reform-oriented mathematics curriculum of Trinidad and Tobago.

GCN tasks are frequently used in textbooks and on examinations in Taiwan (Heinze, Cheng, & Yang, 2004). In the same manner, my findings indicate that the similar task labeled GCNE is used in two of the textbooks in this study. In particular at a higher proportion in MCC. A major feature of the GCNE tasks found in the textbooks, is an accompanying diagram, which initiates physical and mental processes that could lead to the deductive reasoning about its geometrical properties. When diagrams are part of GCNE tasks, they constitute an important

dimension of *cognitive complexity* that requires high-level thought and reasoning of students (Magone, Cai, Silver, & Wang, 1994). By cognitive complexity, I refer to the features of a task that promote students engagement in cognitive processes such as making connections among conceptual ideas and mathematical reasoning (Magone et al., 1994). In this case, the mathematical reasoning refers to the systematic use of mathematical tools to explore patterns, perform geometric calculations, frame problems, and justify reasoning processes (Henningsen & Stein, 1997).

For example, the presence of markings in a given diagram, which indicate parallel or equal sides, could help students deduce geometric facts about these properties that can help solve for missing angles or sides in the diagram. Additionally these markings could guide students in deducing supporting reasons for their actions in the required geometric calculation. Researchers have speculated that the experience of Taiwanese students with GCN tasks is responsible for their superior performance on tasks involving geometric proof in cross-national comparisons (e.g., Heinze et al., 2004). Additionally in the United States, where Geometry instruction has historically focused on developing students' proof and proving competence (Herbst, 2002), GCN tasks are being considered as means to engage student with cognitively complex mathematics (Hsu, 2014). Therefore in considering the similarity between GCN and GCNE tasks, the GCNE-type exercise could potentially promote the type of cognitive complexity that foster students' development of reasoning skills. Hsu and Silver (2014) noted that solving GCN tasks requires students to reason with and about the relationships between the givens and the unknowns, as well as to identify geometric properties that support their problem-solving actions. Due to the strong similarity with GCN tasks, one can conclude that the problem-solving process of GCNE tasks could dictate students' use of algebraic procedures with connections to geometric properties and

theorems and could involve multiple-step reasoning, characteristic of highly complex cognitive tasks.

With regard to the role of Geometry, MCC advocates this content area as one that allows students to experience the work of mathematicians. This is evident by the exploratory class activities that promoted conjecturing and developing deductive arguments. In this textbook, students are given original proof problems so that they can experience the work of mathematicians. Because I coded all GCNE task as developing a rationale or non-proof argument, MCC afforded the most opportunities for student to engage in or reflect on constructing this type of argument. Nonetheless, the GCNE task coupled with the moderate percentage of constructing proof tasks allowed students to apply deductive reasoning through the study of geometric concepts.

The aforementioned characteristics of MCC align with descriptions of Geometry and proof in the competing *mathematical argument* presented by González and Herbst (2002) for the Geometry course in secondary school mathematics curriculum. The mathematical argument promotes Geometry as a school topic where students engage in the creative activities surrounding the work of mathematicians. These activities include pattern identification and conjecturing, which eventually culminate in the writing of proof arguments in support of claims or hypotheses that are made. In this case, proof is considered an important tool for helping students understand the geometrical concepts, thus going beyond the idea of being an exercise on logic. Within the mathematical argument, the Geometry course should offer authentic opportunities for students to develop mathematical ideas. The development of mathematical ideas should mirror the work of mathematicians in their quest to make and validate hypotheses they make based on experimentation and pattern identification. MCC promotes such activity

through the class activities and also allows students to be trained in the activity of pattern identification, conjecturing, and making non-proof arguments. The latter is useful for helping students understand geometrical concepts as they apply their knowledge to write supporting reasons for calculations in GCNE exercises. Overall there exist characteristics of MCC that align with the mathematical model argument for the Geometry course in secondary school.

Mathematics for CSEC

In Mathematics for CSEC (MCSEC), 32% of its exercises offered opportunities for reasoning and proof. The authors highlighted all aspects of the reasoning and proof process in these exercises. In fact, there was a somewhat even split of the percentage of exercises affording students to construct proof and develop rationales (GCNE) type exercises at 43% and 48% respectively. With less than 5% offering opportunities to make and investigate conjectures and 1% promoting pattern identification. Although the trajectory of the processes of reasoning and proof was not offered in one exercise, this textbook to some extent afforded the scaffolding from pattern identification to conjecturing and writing proofs. This characteristic is important with regard to addressing the reformer's vision for the teaching and learning of reasoning and proof in Trinidad and Tobago.

According to policy documents of Trinidad and Tobago secondary school students should be given opportunities to engage in pattern identification, conjecturing, formulating proof arguments and non-proof arguments throughout their mathematical experiences (Trinidad and Tobago, 2005, 2009). Textbooks are important in achieving this objective because they determine when and how students are introduced to reasoning and proof in their mathematical experiences. Furthermore textbooks play a role in providing the support needed for instruction and learning of reasoning and proof (Davis, 2012; Fujita & Jones, 2014; Otten et al., 2014;

Stylianides, 2009; Thompson, et al., 2012). Additionally, the most recent CSEC mathematics syllabus also states that students should engage in the practice of constructing reasonable arguments and critiquing the reasoning of others. The underlying objective of the policy makers' vision is for students to have opportunities for reasoning and proof that may lead to the development of deeper conceptual understanding of mathematical content (de Villiers, 1999; Dreyfus, 1999, Stylianides, 2009) and their understanding of the necessity of reasoning and proof in mathematics (Lampert, 1990; NCTM, 2000). Therefore the inclusion of the all activities of reasoning and proof in the student exercises of MCSEC could provide the support and the type of mathematical experiences for students envisioned by the reformers in Trinidad and Tobago.

In my analysis, I considered the type of justifications the authors used in the various textbooks. Of the three textbooks, MCSEC offered a variation of justifications types. Besides the use of a few confirming examples (empirical) and proof or non-proof arguments (deductive) the authors used alternative type of justifications. The variation of justifications included outlines of proof arguments, referencing proofs in past or future chapters, and left justifications to be completed by the student. The latter case is important because the authors gave students the autonomy to develop their own arguments but providing sufficient guidance to not undermine their thinking or leave them struggling. Within MCSEC, the authors' guidance was located in highlighted commentaries in both the textbook exposition and student exercises. These commentaries usually provided hints and guidelines to support students' thinking in the cases of left-to-the-student or outline justifications. This unique feature demonstrates the authors' emphasis on guiding students sufficiently so that they can adequately attempt tasks provided throughout the textbook's chapters. Davis (2012) claimed that the role of textbooks should not be limited to being an authority on mathematical knowledge but also for scaffolding students during

the teaching and learning of reasoning and proof. Therefore MCSEC has the potential through the authors' commentaries to assist teachers during instruction by providing the guidance typical of an instructors' edition of mathematics textbook. Additionally these commentaries can support students thinking when doing reasoning and proof tasks.

MCSEC also provided extensive opportunities for developing a rationale (RationaleDev) and constructing proofs. However similar to MCC developing a rationale outweighed the opportunities for constructing proof arguments. A possible explanation could be the occurrence of GCNE type of exercises in MCSEC, which promoted developing rationales or non-proof arguments. With regard to the prominence of RationaleDev exercises, one might argue that an "Explain your statements" prompts in these exercises may afford students the opportunity to possibly construct a proof. This is because such exercises promote the explanatory role of proof (de Villiers, 1995; Hanna, 1995). However one could argue that students may not realize that a proof would be a possible response to a GCNE or RationaleDev exercises. Herbst and Brach (2006) found that Geometry students in the United States considered proof tasks and explanation tasks to be distinct. Therefore it is possible that the authors' choice to use the phrase "explain your statements or reasoning" instead of the word "prove" may induce a somewhat implicit nature of the expected argument for these exercises. This begs one to consider beyond this study the implications of students' conceptions of whether a response to an explanation task such GCNE exercise would be considered a proof.

Overall, MCSEC promotes all the activities of reasoning and proof, adhering to the competing *mathematical argument* presented by González and Herbst (2006). However a distinguishing feature of this textbook was the use investigative activities, which promoted everyday Geometry situations to investigate geometrical concepts. The connection of Geometry

to non-mathematical careers suggests the utilitarian role of Geometry in students' future experiences. This feature aligns with the competing *utilitarian argument* proposed by González and Herbst (2006). As a result MCSEC promotes Geometry as a topic whereby students can engage in work of mathematicians and also see the usefulness and connections of mathematics to everyday life or future careers. For example the authors' provide contextual worked examples and exercises that are framed within construction, architecture and science. By using these contexts, students view the usefulness of geometric concepts in applications associated with career-oriented activities in these fields. The authors relate geometric concepts and formulas to solve problems in job situations or to model real-world scenarios. As a result students are motivated to study Geometry applications to work situations.

The findings of my analysis of the type of mathematical statements highlighted that the textbooks' exercises emphasized particular statements or a general statements with a particular instantiation provided. This emphasis has implications for students learning of reasoning and proof. In such exercises, I argue, in agreement with other researchers, that these exercises do not necessitate deductive reasoning to the same extent as exercises about general objects or situations (Otten et al., 2014). For example, when students are given a particular diagram of a triangle with a missing measure for one of the interior angles and are asked to prove that the specific angle is say, 60^0. A student's response may be similar to their response to class activities in MCC or explorations in MCSEC. The student may think it is reasonable to approach this problem by using a protractor to measure the angle in question. As noted by Otten and colleagues (2014), such an approach may seem practical in a real world situation. In this case the more acceptable and expected action by students would be the use of the general idea that the sum of interior angles of any triangle is 180^0. By using this general assertion students can use the

given interior angles to create an algebraic expression and thus solve for the missing angles in the particular case. Reasoning with assertions about the interior angles of any triangle helps students deduce arguments that support their problem-solving actions. However, because particular statements in student exercises do not necessitate deductive forms of reasoning, they have the potential to give students the idea that empirical forms of reasoning are acceptable when doing proof tasks.

Although reasoning and proving around particular statements may not necessitate deduction to the same extent as general statements, they may provide rich opportunities for teachers to help illuminate the general in the particular (Mason & Pimm, 1984) and "think big" even as they are "talking small" (Cuoco, Goldenberg, & Mark, 1996, p. 384). Particular statements can help in creating meaningful shifts to more general-based types of reasoning. In fact, Otten and colleagues (2014) noted that the prevalence of general with particular instantiation statements in student exercises in U.S. textbooks is useful to both teachers and students. In the same manner, I concur that the prevalence of these exercises in textbooks in Trinidad and Tobago could help teachers facilitate the transition from inductive reasoning to deductive reasoning. By focusing on an example representative of a larger class of objects, teachers can help students develop arguments that hold for an entire class or objects. By observing patterns or properties in this selected example, students can make and test conjectures. These activities could motivate students to eventually write a generalized proof argument that holds for the entire class or objects. Therefore teachers can enact such opportunities surrounding general with particular instantiation in ways that highlight the utility of selecting an arbitrary element with which to reason about an entire set and motivate students' engagement in reasoning and proof.

Conclusion

In conclusion, this study makes primary contributions to the research on reasoning and proof in school mathematics textbooks in Trinidad and Tobago. The characterization of the quantity and quality of reasoning and proof opportunities in the Geometry sections of the textbooks can help policy makers, teachers, and all educational stakeholders understand the nature and extent of the existing opportunities for reasoning and proof in the curriculum materials. This is important because it will help educators understand the type of support afforded to teachers as they develop lessons promoting reasoning and proof. Additionally, by understanding the nature of the reasoning and proof opportunities, mathematics educators can determine whether teacher have the support needed for meeting recommendations for reasoning and proof in policy documents and the reformers' vision. In the case of students, it is important to understand the type of support and guidance the textbooks provide for students' preparation for reasoning and proof items in the CSEC mathematics examination.

Overall my findings suggest apparent similarities between Trinidad and Tobago textbooks and U.S. textbooks analyzed by Otten and colleagues (2014). These include, (a) the exemplification of inductive reasoning though explorative activities or empirical justifications in the textbook expositions (b) the high expectation of deductive reasoning in the student exercises, (c) promotion of general with particular type of statements and justifications, and (d) authors' use of fill-in-the-blank, left-to-student, and outline justifications. More importantly, my findings highlight some unique characteristics such as: (a) the promotion of the explanatory role of proof in the GCNE exercises and (b) the scaffolding of proof construction through activities promoting pattern identification and conjecturing and (c) the varying advocacy for Geometry as a content area promoting the work of mathematicians and (d) the utilitarian purpose of mathematics in the

real world. These aforementioned attributes seem to align with reformer's vision for the necessity of reasoning and proof in secondary school students' mathematical experiences in Trinidad and Tobago. Further studies are needed to examine how teachers use these opportunities in the textbooks to support their teaching of reasoning and proof. Additionally, investigate whether students view informal mathematical experiences such as providing explanations to support mathematical claims could lead to constructing proof arguments. The findings of this study can potentially help in guiding the design of future textbooks and instructional resources, which offer opportunities for students to see the intellectual need of reasoning and proof in their mathematical experiences. For example, authors could have initial activities that allow students to engage in pattern identification to conjecturing in all topics. Through these activities, students could develop their inductive reasoning skills before moving on to deductive reasoning activities. Additionally, conjecturing allows students to have the autonomy to be sources of mathematical ideas. Furthermore students will see the need for validating their proposed conjectures. As a result students will see the necessity of supporting ones claims with a reasoned explanation, which has the potential to lead to constructing a proof. The affordance of all activities of reasoning and proof could help students in seeing the explanatory role of proof and the necessity of proof in their school experiences.

CHAPTER 3:

THE TEACHING OF REASONING AND PROOF IN TRINIDAD AND TOBAGO

Introduction

Teachers are the most important agents of instructional policy (Borko & Putnam, 1998) Cohen, 1990, 2011; Cohen & Hill, 2000; Cuban, 1990). Teachers' views of mathematical knowledge, their perspectives of students learning, their use of curriculum materials, classroom practices, and pedagogical decisions are influential in their agency. Cohen and Hill (2000) acknowledged this agency by noting that a teacher's pedagogical practice is an intermediate dependent measure of policy enactment and a direct influence on students' performance. Therefore teachers are a key connection between policy and practice. The ideas that guide this study are grounded in theories about teachers' responsibilities in creating reform-oriented classroom environments that foster the development of reasoning and proof skills, and social constructivist perspectives of teaching and learning. Teachers have a pivotal role in promoting a classroom environment that advances a social constructivist perspective of teaching and learning mathematics. Furthermore, teachers' interactions with textbooks and assessment materials may possibly influence the type of mathematical practices they establish in their classroom and their response to the reformers' recommendations for classroom instruction.

The aim of this study is to understand the classroom microculture associated with the teaching of reasoning and proof in secondary schools in Trinidad and Tobago. I conducted classroom observations and teacher interviews to create descriptive studies of three teachers' instruction of proof at selected school sites. With this study, I intend to contribute to current discussions regarding the links between policy, theory, and practice, considering the specific case of increasing opportunities to engage students in reasoning and proving activities.

Research Questions

The main research questions, which guide my inquiry, are:

RQ.1 How do teachers implement proof-based geometry mathematical tasks in their classrooms?

RQ.2 What sociomathematical norms do teachers establish during the teaching of reasoning and proof?

RQ.3 How do teachers use curricular materials (e.g., textbooks and exam preparation materials) to support the teaching of reasoning and proof?

The three research questions are important to my inquiry because they each address specific possible components of the teachers' instructional practices. The first question pertains to the teachers' pedagogical choices, which includes their choices of reasoning and proof tasks, their instructional strategies, their expectations about students' ability to construct proof, and their daily routine when teaching proof-based content in Geometry. The second question helps in describing the sociomathematical norms teachers establish during the teaching of reasoning and proof. By sociomathematical norms, I refer to Yackel and Cobb's (1996) definition of the social norms teachers establish in their classrooms so that students can work and learn together. These norms are particularly important for supporting students' mathematical thinking. The second question in this study addresses this notion. This question is important because it helps in identifying unique mathematical customs that are culturally situated with doing proofs in Trinidad and Tobago. The third research question helps in identifying the role of the mathematical content in the textbooks and Caribbean Secondary Education Certificate (CSEC) examinations preparatory materials in teachers' decision-making and their construction of opportunities for reasoning and proof. Overall each of these questions is important in guiding my

description of the classroom nature of the teaching of reasoning and proof in Trinidad and Tobago.

Context of Study

Trinidad and Tobago's reform of mathematics education resulted in the creation and implementation of the new national mathematics curriculum for lower secondary school (forms 1 to 3) in 2003 and upper secondary school (forms 4 and 5) in 2009. Along with the reform recommendations, the external examiners, Caribbean Examination Council (CXC) revised the mathematics examinations syllabus to reflect the new curriculum. Additionally the central governing body the Ministry of Education (MOE) revised the authorized textbooks for mathematics instruction.

In addition to the changes in curriculum materials, there are new recommendations for instruction. Teachers are required to adopt a new social constructivist approach to teaching and learning. The implications of such an approach is that teachers must act as facilitators or guides in helping students construct the required knowledge from their pre-existing ideas. With a constructivist perspective, teachers would need to generate situations that stimulate students' mathematical activity and realize that substantive learning occurs through interaction, conflict, and surprise (Wood, 1999). Thus, teachers in Trinidad and Tobago will need to be familiar with practices that support a constructivist perspective to student learning.

Past studies, conducted prior to the implementation and full adoption of the reform policies, described typical instruction in mathematics classrooms in Trinidad and Tobago as "rote teaching" where the emphasis is on drill and practice, rehearsed algorithms, the memorization of facts and formulae, and the regurgitation of rules for success on CSEC examinations (Bell-Hutchinson 2004; London, 1997). Students were not typically encouraged to

question or explore mathematics through open-ended situations. Bell-Hutchinson (2004) also claimed that any mathematical knowledge that students acquire tends to remain "fragile" and inhibits their use of the knowledge outside of the exam-oriented goal within which that knowledge was taught to students. The evidence of the "fragility" of the knowledge gained is seen in annual CXC reports on students' performance (CXC, 2013, 2014). Given that there are no existing studies on mathematics instruction after the implementation of the reform in Trinidad and Tobago, the findings of this study will potentially help researchers understand (a) how the recent reform policies are influencing current teaching practices and (b) how the reform policies may impact teachers' pedagogical decisions and their establishment of sociomathematical norms during the teaching of proof. By examining teachers' instructional practices, the findings of this descriptive study could also help inform researchers, policy makers, and educational stakeholders about any challenges teachers may face when teaching reasoning and proof.

The reform recommendations for classroom practices. According to the new national mathematics curriculum, teaching strategies should represent problem-solving situations in which students are encouraged to collaborate with one another through discussion, debate, elaborate, and clarify their ideas in order to negotiate meaning and understanding of mathematical concepts and principles (Republic of Trinidad and Tobago, Ministry of Education, 2009). Teachers must offer students opportunities to construct, derive, gather and present information, reflect, give reasons, justify, suggest, design, provide examples, and pose questions while learning content areas such as Geometry. All of these pedagogical expectations describe elements of a social constructivist perspective on teaching and learning.

The social constructivist perspective posits that knowledge is shaped by the cultural influences and developed through increasing participation within different communities of

practice (Cobb, 1994; Vygotsky, 1978). Students' learning is directly linked to their social communication with knowledgeable others (teachers, textbooks, and other students). Furthermore, the reform documents include instructional guidance. For example "Classroom activities and student assignments must provide opportunities for students to communicate, reason mathematically, make connections, and solve problems" (Republic of Trinidad and Tobago, Ministry of Education, 2003, § 2:3). This quote suggests that the reformers' vision for classroom instruction highlights the importance of social interaction as an integral component of classroom instruction during problem solving.

For example, the Geometry unit of the curriculum adopts a social constructivist orientation (Republic of Trinidad and Tobago, Ministry of Education, 2003). The aim of this unit is to make the learning of mathematics meaningful and relevant for the student. According to the authors, "teachers must provide a learning environment where students search for [create] meaning, appreciate uncertainty, inquire responsibly, and discuss their ideas" (Republic of Trinidad and Tobago Ministry of Education, 2003, § 1:3). This type of learning environment fosters the development of mathematical literacy, problem-solving skills, and reasoning ability in students from a social constructivist perspective. Accordingly, the authors provide suggested lesson plans with problem solving activities for various Geometry topics.

Overall, the new mathematics curriculum calls for more mathematically engaging work for students, and for teachers to help students understand math rather than memorizing facts and operations. The curriculum writers have advocated for the adoption of a constructivist approach to teaching whether teachers act as guides or facilitators of students' construction of their experiential knowledge. The curriculum documents include encouragement for teachers to (a) to make students more active participants in classroom discourse, (b) pay more attention to

students' mathematical ideas, and (c) place much more emphasis on mathematical reasoning and explanation rather than the mathematical facts and skills. Therefore, teachers whose actions are more aligned with behaviorists learning theories may find that the current recommendations conflict with their instructional practices. This study investigates the nature of the classroom instruction surrounding reasoning and proof. Therefore the description of teachers' pedagogical practices can help researchers and the reformers understand the interplay between the expectations for reform-oriented teaching and what actually takes place in the classrooms. As a result, the findings of this study could add to existing dialogues between policy and practice in Trinidad and Tobago.

Literature Review

In any reform movement there are possible implications as teachers try to implement the reform recommendations in their classroom. As in the case of Trinidad and Tobago the emphasis on the adoption of a social constructivist perspective has implications for teachers' pedagogical decisions, classroom practices, and expectations of students. First, I discuss the pertinent studies that help in my examination of reform implications for teacher's instruction. Second, I discuss specifically reform-oriented teaching with regard to the instruction of reasoning and proof. This will help in understanding the documented findings about how teaching practices that motivated students to engage in reasoning and proof interact with reform initiatives. Third, I discuss teachers' use of curriculum materials such as reform-oriented textbooks. Finally, I close by demonstrating how the aforementioned three previous components contribute to my overarching inquiry as it relates specifically to Trinidad and Tobago.

The Implications of Reform on Teaching and Learning

In this study, I examine the case of reform framed within a centralized educational system and its impact on teachers' instruction. Past studies have indicated the dynamic relationship between reform-oriented policies and the actual instruction in classrooms. Several researchers have looked at the interplay of the reformer's vision with teachers meeting the demands during instruction in California, China, UK, Japan, and Ireland (e.g., Cohen & Hill, 2000; Hoyles, 1997; Jones, Fujita, & Ding, 2006; Lubienski, 2011). In considering the implications of the reform efforts on teachers' practice, it is worthwhile considering whether the various changes would lead to a case of policy impacting practice or a case of teachers' practices impacting policy. Past studies have shown that teachers adopt new instructional policies through the lens of their old ingrained pedagogy (e.g., Cohen, 1990; Cohen & Hill, 2000; Cuban, 1984). Cohen and Hill (2000) noted that reform-oriented policies usually seek to increase student achievement by manipulating elements of instruction including assessments, curriculum materials, teachers' knowledge, and classroom practices. However adopting new instructional approaches takes times and may not be easily attainable (Kennedy, 2005; Darling-Hammond, 2008; Lubienski, 2011), because teachers' prior pedagogical practices, treatment of knowledge, and experiences influences their implementation of reform-oriented instructional policies.

Mixing of New and Old Teaching Practices

Several studies have shown the dynamic relationship between reform-oriented instructional policies and teaching practices (e.g., Cohen; 1990, Cohen & Hill, 2000; Cuban, 1984; Kennedy, 2005). These studies claim that teachers' past experiences influence their adoption of reform-oriented instructional policies. For example, Cuban (1984), named teachers mixing of traditional and new practices as they attempt to adopt new instructional policy as

"conservative progressivism." Cuban described this as the approach of teachers in responding to instructional policies whereby they progressively construct hybrids of particular new practices that seem to be best suited to what they ordinarily do in their classrooms. If components of teachers' past practices were most applicable to their school environment and students learning abilities, teachers seldom abandon them. Teachers would gradually adopt the new practices that either provide little disruption to their familiar ways of teaching or helps them to continue meeting the needs of their students and school.

Kennedy (2005) supports Cuban's argument based on her study with 45 teachers from several states in the United States. Kennedy observed and interviewed the teachers about their instructional practices with regard to recent reform initiatives at their schools. Kennedy claimed that as teachers tried to implement instructional policies they considered their old practices that have worked successfully in the past. As a result, teachers try their best to keep these old practices, thus they add minor improvements so as to meet the reformers' vision. In the end their teaching is simply a mixture of old and new teaching practices. Kennedy states that this mixing highlights the reality of teachers attempting to bridge reformer's ideal with everyday teaching practices.

Teachers' Treatment of Mathematical Knowledge

Another influence on teachers' adoption of reformers' instructional policies is the teacher's view of knowledge. In the case of teaching mathematics, Cohen (1990) examined the 1980's reform of mathematics instruction in California. The reformers' vision for instruction included a more inquiry-based style of teaching rather than direct instruction. Cohen described the case of a teacher's past pedagogical practices influencing the instructional policy. Cohen presented an example of a teacher, Mrs. Oublier, who blended reform recommendations

supporting inquiry-based teaching with her traditional view of mathematics as a fixed set of practices. Mrs. Oublier ‘s didactic style of teaching lesson did not foster students’ exploration of mathematical ideas through inquiry-based activities. Cohen noted that as Mrs. Oublier tried to find her way from familiar to new teaching practices she mixed the new ideas onto the old practices. Therefore her teaching demonstrated a mixture of a new inquiry-based philosophy for learning with her view of mathematical knowledge and pedagogical practices. Although, Mrs. Oubier intended to transform her math teaching, her perspective about mathematical knowledge as fixed impacted how she transformed the written curriculum to the intended and enacted curricula. I find this case study particular useful in understanding how a teacher’s past conceptions about the treatment of knowledge influences their use, and in this case how Mrs. Oubier followed, planned and enacted the lesson in the reform-oriented textbook.

Cohen (2011) explained further the phenomena of teachers’ treatment of knowledge influences their response to reform-based instructional policies. Cohen explained that the treatment of knowledge provides an example of when teachers’ conceptions about how knowledge should be treated agrees or disagrees with policy recommendations and thus is reflected in how a teacher transforms the written curriculum to the intended and enacted curriculum. Cohen (2011) noted that if teachers treat knowledge as a result of inquiry, then they will “read” the curriculum to obtain opportunities that support developing, contesting, and constructing knowledge with learners. The teachers’ views about how knowledge should be treated whether as through inquiry or as a fixed set of rules will influence how they integrate the instructional policies into their classroom practices. In Trinidad and Tobago, teachers may adopt the new instructional policies through the lens of their old ingrained pedagogy that stressed rote-memorization, procedural fluency, didactic teaching, individual work, and highly focused

recitation. Therefore it is worthwhile considering whether teachers in Trinidad and Tobago may adopt the reformers' vision in light of what they know and what they deem as feasible within the school environment, governing authorities, textbooks, and external examinations.

Reform and Teaching of Reasoning and Proof

Reformers in different countries have attempted through various initiatives to alter instruction of reasoning and proof (e.g., China, Japan, United Kingdom, Germany & France). Several researchers examined the interaction between these implemented reform initiatives and the teaching and learning of reasoning and proof (e.g., Ding & Jones, 2006; Hoyles, 1997; Jones, Fujita, & Ding, 2006). In some cases, the reform-oriented curriculum shaped instruction and in other cases, ingrained culturally based notions coupled with the reformers' vision for learning influenced the teaching of reasoning and proof.

Hoyles (1997) coined the phrase curricular shaping to describe the ways in which the reform-oriented curriculum influences and shape students competencies with proof. Hoyle described how the national curriculum and reform-oriented textbooks in the United Kingdom, motivated teachers to engage students in explorative activities of pattern identification, conjecturing, and constructing proofs. The curricula materials shaped the content and instruction of proof. Therefore the teachers, as agents of the reform, adopted practices that helped students (a) engage in data generation, pattern recognition and conjecturing (a) have more opportunities to develop inductive rather than deductive reasoning skills and (c) develop reasoned explanations of why conjectures are true rather than confirming they are true, and (d) engage in social argumentation. The latter refers to students' use of mathematical language, symbols and everyday language to effectively present a convincing argument during class discussions. Hoyles claimed that the curriculum also shaped students' conceptions of proof and proving.

Ding and Jones (2006) found that teachers in response to reform initiatives in China, used questioning as a means of engaging students in reasoning and proof. Jones and colleagues reported findings based on observational data of three Grade 7 teachers instruction of angles in parallel lines in China. Ding and Jones (2006) identified that there were culturally specific teacher practices. For example, in China, a distinctive character of Confucian heritage with regard to learning is to ask questions continuously and to review one's prior knowledge regularly. This cultural perspective was reflected in the ways teachers changed their teaching strategies to achieve the reform initiatives to make mathematics more inquiry based. Jones and colleagues found that the teacher's questions were carefully sequenced and motivated students to discuss definitions and theorems. As students gained the autonomy to question each other's proposed definitions and conjectures, the teacher carefully introduced special vocabulary to help students articulate their ideas. As a result of these strategies, the students gradually became involved in examining the characteristics of each definition and theorems associated with angels and parallel lines. The teacher also used instructional strategies such as open-ended questions and group work to promote the discovery role of proof. The teacher played an integral role in enhancing students' thinking as they transition from methods of inductive to deductive reasoning.

Ding and Jones (2006) reported that the instructional model teacher used for teaching the new geometrical concepts included, (a) empirical investigations to observe, guess, or verify a geometrical property; or pose a question or problem to initiate inductive reasoning, (b) draw figures to represent proposed conjectures or hypotheses (c) test and prove the conjecture's validity (d) use accurate mathematical language to present the established fact or theorem and show the basic figure representative of the established result and (e) read and highlight the key

words in the theorem. Ding and Jones noted that the aforementioned pedagogical approaches were reflective of the reform initiatives to exemplify the role of reasoning and proof in lower secondary school. The findings of this study suggest that the teachers seemed to meet the reform initiatives by employing culturally based strategies of questioning and group work in their teaching of reasoning and proof.

Jones, Fujita and Ding (2006) also examined the relationship between the reform initiatives and the teaching of reasoning and proof in Japan. Jones and colleagues conducted a case study of a Grade 8 teacher who adopted the national instructional initiative "Lesson Study." Lesson Study is a governmental initiative designed for professional development of teachers in Japan. This initiative involved the collaboration of teachers working in teams to create lesson plans through several cycles of planning, teaching, and reviewing (Fernandez & Yoshida, 2012). Jones et al., (2006) reported that a teacher would typically use two methods for teaching reasoning and proof according to the Lesson Study initiative. In the first approach, teachers would typically provide a proof-based problem requiring students to write a proof for a stated geometrical result. For example "Let us prove that if $\overline{PQ}$ is parallel to $\overline{BC}$ in triangle ABC, then triangles *ABC* and *APQ* are similar." In the second stage, students work in groups to construct a proof. In the finally stage the various groups shared there proof for whole class discussion. In the second approach the teacher would initially give students a more challenging construction problem. For example "Let us consider how to trisect (divide into three equal parts) a given straight line $\overline{AB}$." In this approach students are encouraged to consider why their constructions would work. Additionally, students discover theorems and geometrical properties through the construction activity. After students discuss their constructions in groups, the teacher would introduce the following theorem: In a triangle *ABC*, *P* and *Q* are located on the lines $\overline{AB}$ and

$\overline{AC}$ respectively. If $\overline{PQ}$ is parallel to $\overline{BC}$ then $AP: AB = AQ: AC$ and $AP: PB = AQ: AC$, which students noticed and discussed during their group discussions about their constructions. The teacher solicited explanations supporting the relationship between students' constructions and the resulting theorem. In both approaches the teachers asked questions and facilitated students reasoning and proving geometrical results.

The aforementioned studies demonstrate that in these countries, teachers' response to reform recommendations included pedagogical approaches that assisted their students in creating definitions and conjectures, and eventually constructing proofs. In the three educational situations, the reform initiatives promoted teachers' adoption of teaching strategies that provided opportunities for students' engagement in reasoning and proof. These studies are useful for my inquiry because similar to Trinidad and Tobago, China Japan, and United Kingdom, have a centralized governance of education. Each of their respective instructional initiatives offered uniform prescriptive guidance through national policies, national curriculum, national examinations, and textbooks. Due to the similarity of the educational context, these studies provide documented evidence of the possible implications of reform-based policies on teachers' instruction of reasoning and proof.

Knippings (2008) examined the classroom practices surrounding the teaching of proof in six French and six Germany grade 8 classes. Knipping's analysis of observation data of the teaching of Geometry suggested that the difference in cultural classroom norms highlighted differences in proving processes. In each country, the teachers established classroom practices that influenced the types of argumentation students used when proving. For example in one type of argumentation, the teacher encourages the students to make conjectures, which are examined together in whole class discussion. Students' conjectures are valued even when they are publicly

refuted and contested. In another culture, arguments must be valid before they are shared with the class. In the same manner I hypothesize that there may be unique cultural practices that govern the type of opportunities teachers create for students engagement with proof in Trinidad and Tobago. This study contributes to the current discussions about the teaching of proof as it relates to an international context. As I stated earlier Trinidad and Tobago provides a case of the bigger issue of the implications of reform in a centralized governing education system on the teaching of proof. There are no studies examining these phenomenon post-reform efforts in Trinidad and Tobago. Therefore the findings could provide an understanding how the curriculum shapes instruction and cultural influences on the instruction of proof.

Social Constructivism and the Teaching of Proof

In the recent reform of Trinidad and Tobago, reformers advocate for a social constructivist approach to the teaching of reasoning and proof. The new curriculum advocates for increased social interaction between teachers and students as they engage in the construction of knowledge in an inquiry-based mathematics classrooms. Several researchers noticed that in order to participate successfully in a social-constructivist environment, students' participation must shift from a passive to an active role (e.g., Wood, Cobb & Yackel, 1991). In several studies conducted by mathematics education researchers (e.g., Cobb et al., 1991; Wood et al., 1991), found that teachers needed to shift the behaviors of the students as early as first to third grades, to encourage them to listen to other students and to talk about their solutions. Drawing on the work of Bauersfeld (1988) and Voigt (1985), Cobb et al. (1991) claimed that the teacher and students jointly constructed what constitutes knowing and doing mathematics. Cobb and colleagues asserted that the ways of explaining and justifying one's ideas are socially accomplished as the teacher and students engage in discussions. As a result, teacher and students mutually established

certain norms for engaging in mathematics discussions. These norms are called "sociomathematical norms" (Yackel & Cobb, 1996) and include what counts as a different mathematical explanation, an acceptable mathematical explanation and a sophisticated mathematical explanation. Cobb et al. (1991) concluded that the creation of these sociomathematical norms provided the social context whereby learning can take place. Additionally these norms affected the type of learning that took place in the classrooms they studied. The evident changes in learning included: different conceptual ideas were developed and shared, also students' reflection on their own thinking were strengthened.

Adopting a social constructivist perspective can change teachers' actions and intentions. The changes can occur in the nature of classroom activities, the role of the teacher, the sociomathematical norms teachers establish, and their pedagogical decisions. The critical role of the teacher while using a social constructivist perspective was captured in an exemplary study of the culture of a mathematics classroom, conducted by Cobb et al. (1991). In this study, the researchers explored the analogies between scientific communities and the social life in a second grade classroom in mathematics. Their work revealed how the teacher created a classroom where the children were evaluators of one another's ideas, including establishing norms such as persisting in the solution of personally challenging problems, explaining personal solutions to one's partner, listening to and making sense of the partner's explanation and attempting to achieve consensus about the answer, and a solution process. By the end of five months, these norms were in place, and the teacher had to do less to guide children toward these norms.

Sociomathematical Norms and the Teaching of Proof

According to Martin and McCrone (2003), teachers' actions significantly impact classroom discourse as well as students' opportunities for learning how to construct proofs.

Martin, McCrone, Bower, and Dindyal (2005) discussed the relationship of teacher and student actions in the teaching of proof in Geometry. They claimed that a teacher's decisions not only influences classroom interactions but also the expectations of students and teachers particularly during proof-related activities. In their study, they examined four classroom episodes, which involved the teaching of geometrical proofs. Through these episodes, they examined the effect of teachers' actions and their pedagogical choices on the students' individual and collective understanding within a social constructivist environment. In their analysis, they identified some teacher actions that helped motivate students to make and prove conjectures. These actions include asking open-ended questions, revoicing students' contributions, using diagrams as strategies to engage discussions, analyzing students' arguments, and placing the responsibility for reasoning on the students. Martin and colleagues also highlighted the importance of establishing classroom customs. For example, students were expected to explain their ideas and provide supporting arguments for their claims during classroom discussions. In this type of classroom environment, students were encouraged to actively participate in classroom discourse that promoted the development of their proof and reasoning skills.

Martin et al. (2005) in their study used the notion of sociomathematical norms to show how teachers' pedagogical choices are essential to the type of classroom environment they establish. The teachers' pedagogical choices also influenced the type of opportunities for reasoning and proof afforded to students. Martin and colleagues argue that in order to foster a classroom climate in which students make and test conjectures, develop reasoned arguments, and construct proofs, the teacher should "engage in dialogue that places responsibility for reasoning on the students, analyze student arguments, and coach students as they reason" (Martin et al., 2005, p. 95). Therefore the teachers' pedagogical choices created a classroom environment

where the teachers and students negotiate the sociomathematical norms, which explains what counts as an acceptable proof.

Teachers' interactions with curriculum materials

In considering the relationships between past reform initiatives and teaching, several researchers have examined teachers' use of curriculum materials in their attempts to meet reform demands. Within any reform movement, curriculum materials such as textbooks can be a mediator between reformer's intentions and what teachers implements in their classrooms. Although textbooks can be important agents of change, teachers using the same required textbooks may interpret and cover the textbook in considerably variable ways. For example, in Cohen and Hill (2000), the researchers observed that teachers' adoption of the textbook to meet reform demands in California. These researchers claimed that teachers adapted the recommended textbooks in varying ways. For example, one teacher followed the textbook's suggestion that the class discuss alternative solutions to problems. This was a new element in the textbook, added in response to the reform policy, The suggestion was intended to get divergent ideas so that students could explore, discuss, and make sense of alternative ideas. However, in following the textbook's advice, the observed teacher conducted a traditional discussion in which the goal was to obtain a single right answer thus frustrating the textbook's intention. This example suggests that as teachers try to use textbooks to support their adoption of new reform-recommended instructional policies practices, resort to traditional more familiar practices that subvert the authors' original intentions.

Lubienski (2011) reported a study of reform efforts in Ireland, which has a similar centralized educational system as Trinidad and Tobago. Lubienski observed the development and implementation of the recent reform movement entitled "Project Math." She noticed that

textbooks functioned as an important mediator between curricular intentions and implementation. Lubienski noted that not all of the so called reform-oriented textbooks would be capable of helping teachers implement the type of instructions advocated by the Project Math policies. For example in her observations of two of these textbooks she found remarkable differences. One textbook highlighted traditional presentation of boxed examples and formulas in the expository sections with worked examples for students to follow. This indicated a single answer response or expected learned procedure for solving problems. This book seemed to advocate the view of mathematics as a fixed set of rules and procedures. However, the other textbook presented a structured sequence of investigative activities to facilitate students' derivation of formulas and proofs. In this case, the textbook seemed to portray mathematics as a result of inquiry, which was more aligned to the objectives of the reform movement in Ireland. Although prescribed textbooks provide some homogeneity in the type of instructional support teachers could receive, teachers have the task of analyzing and selecting textbooks, which help in meeting the demands of reform.

The aforementioned studies are useful in understanding the interplay of teachers' instruction and their use of reform-oriented curriculum materials. In Trinidad and Tobago, teachers are offered prescriptive guidance through the recommended textbooks, national curriculum and CSEC examination materials. As demonstrated in past studies, teachers may use these support materials differently as they attempt to meet the reformers' vision for the teaching of reasoning and proof. Furthermore teachers may also face challenges as they adopt a social constructivist approach to teaching. If teachers are to significantly alter their pedagogy and be agents of reform they must address whether their current practices align with the goals of the reform movement and thus seek ways to adapt their practices if needed to meet the reformers'

demands. Thus, in Trinidad and Tobago, it may possibly be the case of the instructional policies directly influencing practice or the case of teachers' old engrained practices influencing how the policies are enacted in classrooms. The latter case will be reflective of documented studies about the mixing of old and new practices (Cohen, 2000, 2011) or conservative progressivism (Cuban, 1984). The findings of this study will help researchers the dynamics of the instructional policies and the teaching of proof. Additionally these findings will provide another perspective of teachers, creation of opportunities for reasoning and proof framed within a social constructivist environment. The findings can also add to existing literature about reform-based pedagogical approaches for instruction of reasoning and proof. Given the role of the curriculum materials in teachers agency of reform recommendations, this study can potentially lead to further understanding the role of a national curriculum and examinations in teachers instructional response to reform in an international context.

Theoretical Framework

In this study I focused on three areas of classroom practices during instruction of reasoning and proof, (a) classroom microculture, (b) teachers' pedagogical choices, and (c) teachers' use curriculum materials. My theoretical framework for this study takes into account the three aforementioned aspects of classroom practice and their influences on students' opportunities to learn reasoning and proof. The creation of this framework is built on previous work conducted by Martin and McCrone (2003) and Martin et al. (2005), in which they examined the dynamics of the social context of classroom discourse during the teaching of proof.

Figure 3.1 illustrates the components as they relate to my investigation of the factors contributing to teachers' creation and management of opportunities to learn proof. The bi-directional arrow in Figure 3.1indicates the possible interaction between the established

sociomathematical norms and classroom mathematical practices with teachers' pedagogical choices. The one-direction arrows indicate the possible influences of various components on the students' opportunities to learn proof. In the next section I explain how each component of the framework contributes to my overall inquiry

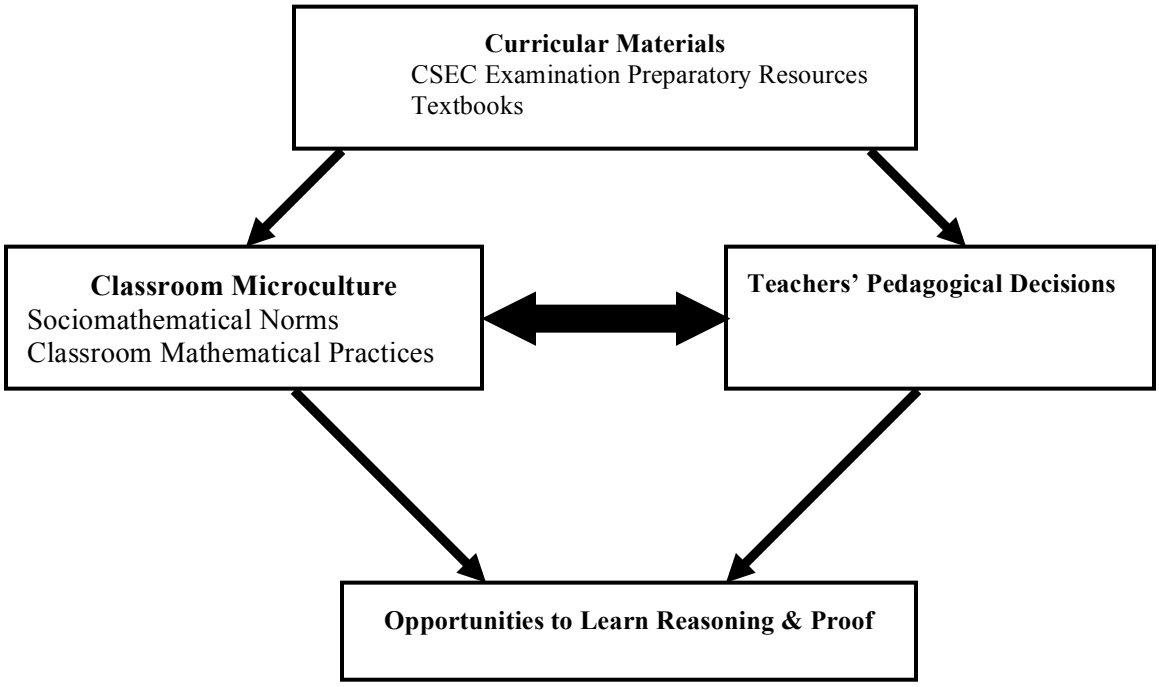


Figure 3.1: The Factors influencing the Opportunities for Reasoning and Proof.

The Components of Classroom Practice

The first component of interest to my theoretical framing is the classroom microculture. This includes the sociomathematical norms and classroom mathematical practices. By sociomathematical norm, I refer to the aspect of the classroom microculture that entails the evolving mathematical customs of the classroom activity specifically associated with proof-related activities (Yackel & Cobb, 1996). For example this will include what count as (a) an

acceptable solution to a proof question, (b) an acceptable mathematical explanation of one's reasoning, (c) a different mathematical explanation of why an argument proves a result, or (d) an acceptable valid proof. By classroom mathematical practices, I refer to the *taken-as-shared* practices that apply to specific mathematics content (Cobb et al., 1991; Cobb & Yackel, 1996; Martin & McCrone, 2003). By taken-as-shared, I mean widely acceptable mathematical practices that are agreed upon by a community of learners such as a classroom. These taken-as-shared practices are specific to the learning of certain mathematics content. For example, a classroom mathematical practice, applicable to my study is the acceptable method students are expected to use to prove the congruency of triangles in Geometry. The two components, sociomathematical norms and classroom mathematical practices are useful in understanding how teachers incorporate a social constructivist perspective to learning in their instruction of reasoning and proof.

The second area of interest includes teachers' pedagogical decisions during the instruction of reasoning and proof. Teachers' pedagogical decisions include (a) the choice of mathematical tasks (b) processes of creating opportunities for reasoning and proof, (c) the instructional strategies teachers use (questioning, revoicing, direct instruction, cooperative learning), and (d) teachers' expectations for students (Martin et al., 2005). My investigation into these four components of teachers' pedagogical decisions are important in understanding how teachers implement reasoning and proof tasks in their classrooms and help in answering my first research question. The sociomathematical norms teachers establish may influence their instructional strategies and their expectations for students. Additionally the classroom mathematical practices may possibly influence the teachers' choice of tasks and methods of creating possible opportunities for reasoning and proof. Thus I consider the influence of the

microculture on teachers' pedagogical decisions. I also consider the possibility that teachers' pedagogical decisions may influence the sociomathematical norms and classroom mathematical practices they establish while teaching. The bi-directional arrow in Figure 3.1represents these two-way influences.

My third area of interest entails teachers' use of curriculum materials such as the textbooks and CSEC examination preparatory materials. In Trinidad and Tobago, the CSEC assessments are considered as high stake because they are used as the benchmark for holding teachers accountable (Bell-Hutchinson 2004; Jackson, 2012; London, 1997.) Additionally, a school's performance on these external exams contributes to their ranking on the MOE's ranked listing of schools (Jackson, 2010, 2012). As a result, teachers may rely heavily on CSEC preparatory materials and syllabi to successfully prepare their students for this high stake external assessment. Because of the possible influence of CSEC examination materials on classroom practices and teachers' adoption of the reform's instructional policy, I consider the CSEC assessments within my third area of interest in this study.

Characteristics of Reform-based Teaching

In this study, I am also interested in observing any evidence of reform-based teaching as defined in the literature. Hufferd-Ackles, Fuson and Sherin (2004) introduced a framework that describes the characteristics of a classroom community enacting reform-based mathematical practices. In particular, the authors presented the development of a *math-talk learning community*. By a math-talk learning community, they referred to a classroom community whereby the teacher and students use classroom discussions to support the mathematical learning of all participants. In such a community the main objective is for each participant to understand and extend their thinking. This objective is similarly proposed in the policies of the recent

mathematics reform in Trinidad and Tobago. As I described earlier, the reformers recommended that teachers must actively engage their students in discussions that promote the development of ideas. Teachers are to guide students thinking so that they can further develop their ideas while engaging in classroom discussions (Republic of Trinidad and Tobago Ministry of Education, 2009). With regard to reasoning and proof, the curriculum writers advocated for the adoption of a social constructivist approach to teaching. Teachers are encouraged to place more emphasis on mathematical reasoning and explanation rather than memorization of mathematical facts and procedures. Teachers should create opportunities for students to share their ideas as well as critique and extend their fellow students' ideas. Students are expected to actively participate and take responsibility for the formation of ideas during classroom discussions (Republic of Trinidad and Tobago Ministry of Education, 2005, 2009). All of the aforementioned recommendations suggest possible changes to both teacher and students' actions during the teaching of reasoning and proof in Trinidad and Tobago.

In their framework, Hufferd-Ackles et al. (2004) looks at the development of the classroom community as the teacher implemented a reform-based curriculum in an elementary school. Hufferd-Ackles and colleagues rated the classroom practices at four levels that indicated the change in teaching practices over a one-year observation. At the highest level of their rating, they described the characteristics of reform-based instruction of mathematics. In particular they described four components of the classroom community as it relates to the application of a social constructivist perspective to the teaching and learning of mathematics. The four components include, (a) the type of questioning, (b) explaining mathematical thinking, (c) sources of mathematical ideas, and (d) responsibility for learning. Figure 3.2 shows the adaptation of these four components within my investigation of the nature of instruction of reasoning and proof in

Trinidad and Tobago. I present this model as the conceptual framework that guided my consideration of the classroom practices as it relates to mathematics reform in Trinidad and Tobago. Because my overarching emphasis of the dissertation study concerns the implications of reform on content, teaching, and student learning, I use the four components labeled in Figure 3.2 to examine the (a) the classroom microculture, (b) teachers' pedagogical decisions, and (c) influence of curriculum materials on instruction.

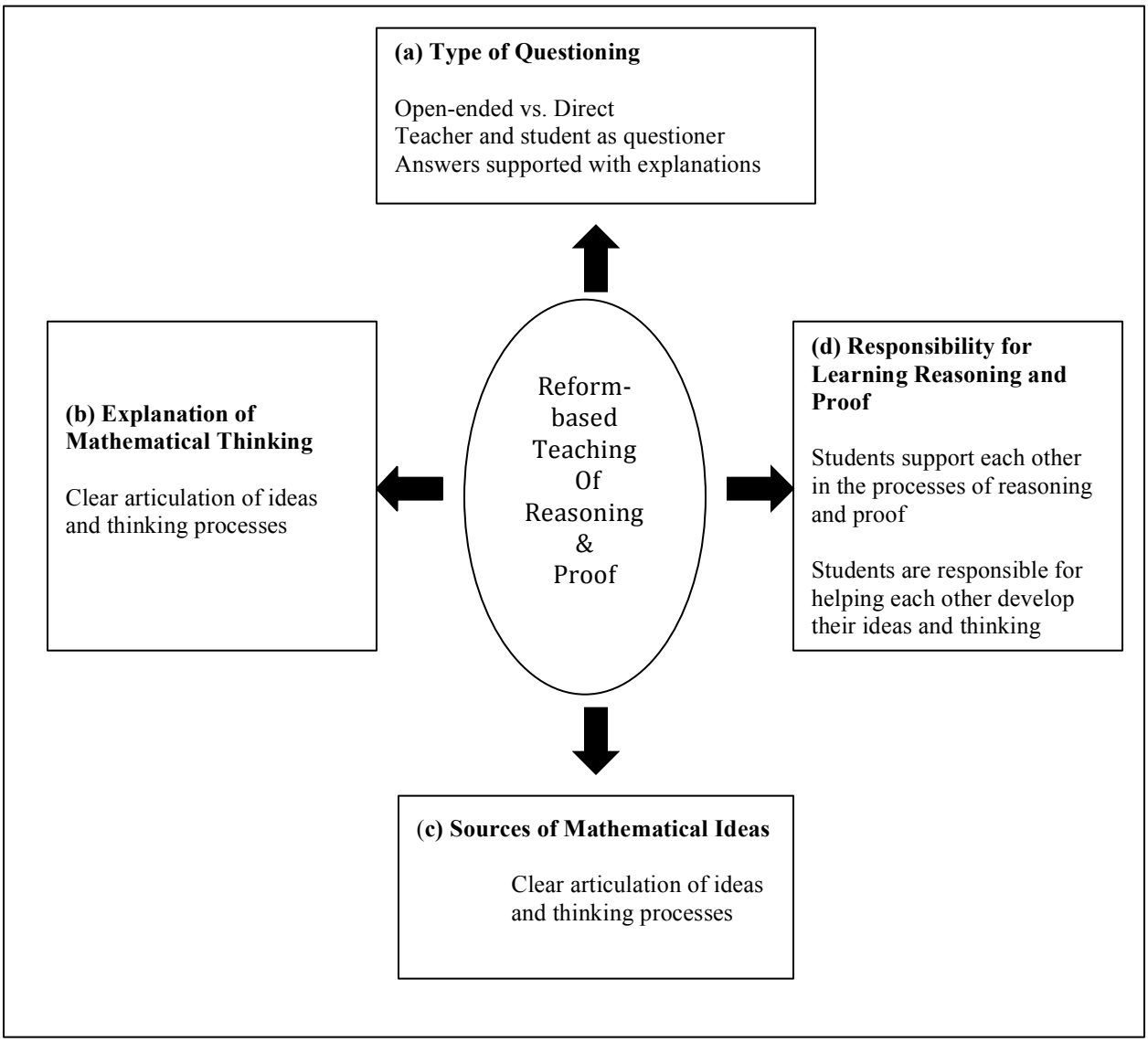


Figure 3.2: Characteristics of reform-based instruction of reasoning and proof. Adapted from Hufferd-Ackles et al. (2004).

The framework helps in framing these aforementioned components within the characteristics of reform-based instruction. I found this framework useful gather evidence that supports the possible implication of the mathematics reform on the instruction of reasoning and proof. In the following section, I discuss each of the characteristics in Figure 3.2 as they relate to my inquiry.

The four components of reform-based teaching of reasoning and proof. The type of questioning refers to the kinds of questions asked and who posed the questions. At the highest level on Hufferd-Ackles et al. (2004) rating of a math-talk learning community, reform-based teaching involved the use of open-ended questions as opposed to direct questions. Open-ended questioning should initiate answers supported by explained reasons. The teacher and students both act as questioners. Teachers ask students questions that prompt their development of ideas, whereas students ask each other questions to support and clarify their mathematical ideas. The explaining of mathematical thinking refers to the level of detail in students' articulation of their ideas during the lesson. Teachers do not expect answer-focused responses but elicit answers that clearly express students' strategies and thinking. This is important because it aligned with teachers' pedagogical choices, where their expectations for students influences the sociomathematical norms created during their instruction.

In reform-based teaching, teachers encourage students to describe their strategies, defend, and justify their answers. At the highest level of Hufferd et al.'s (2004) rating, students are aware of their teachers' expectations and provide supporting explanations with their solutions. Students expect that other students could challenge their solutions so they provide the required explanations that justify and defend their solutions.

The explaining mathematical thinking component in the framework helps in understanding the classroom microculture surrounding students' articulation of their ideas. In a traditional teacher-directed classroom brief answer responses are expected from students. However in a classroom aligned with reform-based instruction, students are expected to discuss their mathematical ideas and reasoning. In accordance with the social constructivist perspective and reform-based instruction described in the framework, students defend and justify their reasoning under the teacher's guidance. The teachers' responsibility is more of monitoring students' thought processes as they discuss ideas with each other. These descriptions help in understanding the extent to which the classroom practices reflect the reformers' ideologies about instruction.

In the third characteristic, sources of mathematical ideas, teachers and students provide mathematical ideas. Students can interject their ideas as the teacher or other students lead classroom discussions. Students are confident that their shared ideas would be valued and possibly provide a catalyst for further discussions or introducing new knowledge. This characteristic supports the idea that within a social constructivist-learning environment, students are also considered as the authority and source of knowledge. The teacher values students' contributions and students respect each other's ideas.

The sources of mathematical ideas component also outlines the role of the teacher in imparting knowledge during discussions. For example, teachers may provide all the ideas for classroom discussions, whereas in a more developed math-talk learning community, the students and teachers share the responsibility of introducing ideas into discussions. Textbooks and other curriculum materials can also be the source of mathematical ideas. This fits into my consideration of the role of curriculum materials such as CSEC preparatory materials in

supporting teachers and student's development of mathematical ideas. Therefore I consider how the teacher's ideas coupled with their use of the curriculum materials influence the direction of the lesson.

The responsibility of learning describes the extent of involvement of students and teachers in constructing knowledge about mathematical content. In the framework, Hufferd-Ackles et al. (2004) described how the responsibility for learning shifts from the teacher to the students. In my study, I pay attention to this component, but as it relates to the classroom mathematical practices and sociomathematical norms that support learning. For example, I adapt this component to look at the responsibility for establishing classroom mathematical practices surrounding the learning of how to prove the congruency of triangles. My reason for the adaptation of this component was due to the time frame I observed the classrooms. My inquiry was unable to fully measure the students learning. As a result I looked at the classroom mathematical practices that could support student's learning of congruency of triangles through the processes of reasoning and proof.

In explaining the different outcomes, associated with applying a social constructivist perspective to teaching, several researchers claim that success with learning through this perspective is a function of the extent to which teachers promoted shared ownership of the learning (Cobb et al. 1991; Taylor & Cox 1997). As a result, the establishment of classroom norms and the implementation of interpersonal activities via group work and discussions support the arbitration of shared knowledge. The incorporation of the Hufferd-Ackles et al. (2004) framework, which describes the components of a math-talk community, helps in understanding the type of mathematical practices, which promote the shared understanding of mathematical knowledge. This is one of the objectives outlined for reasoning and proof in the reform

recommendations in Trinidad and Tobago. Overall, I consider Hufferd-Ackles et al.'s (2004) framework useful for understanding how the classroom norms help promote a social constructivist perspective of reform-based mathematics teaching.

Methodology

In this study I conducted multiple descriptive case studies of three teachers' instruction of a proof-based lesson in Geometry. Multiple case studies are useful when addressing descriptive research questions with multiple units of analysis (Bieda, 2009; Yin, 2006) and for comparisons when investigating educational phenomena (Stake, 1978). Given that the research questions required descriptions of teachers' and students' involvement during the teaching of reasoning and proof, the multiple case study design supported data collection to address the intent of this study. For this study, the case was the teaching of a proof-based lesson of the topic *Congruent Triangles* in the CSEC mathematics syllabus. My selection of this topic was based on findings of a preliminary analysis of the three most commonly used textbooks in Trinidad and Tobago. My analysis indicated that the topic *Congruent Triangles* offered the highest number of opportunities for reasoning and proof. Therefore I selected this topic for the proof-based lesson I observed. Although the lesson was proof-based, I also considered reasoning activities that were associated with the lesson. My inclusion of reasoning facilitated the possibility that the classroom activities may not involve actual construction of geometrical proofs but offer activities that led to pattern identification, conjecturing, and developing non-proof arguments.

School Site Selection

In Trinidad and Tobago students are assigned to secondary schools after fifth grade, based on their performance in the Secondary Entrance Assessment (SEA) examination[11]. As a result of this assignment there exists large differences in the school environments to which students of different entry levels of achievement are assigned (Jackson, 2010, 2012; London, 1989). This assignment has also led to the MOE's ranked listing of secondary schools. The listing is based on (a) entry level ability and score of students in SEA, (b) the schools overall performance at a national level in the CSEC examinations, and (c) the number of national scholarships received by students at the school based on their performance in the Caribbean Advanced Proficiency Examination (CAPE). Additionally students taking the SEA exam, usually list their secondary school choices. The MOE ranking influences students' choices and students usually rank higher-achieving schools higher on their lists. The possibility of a student being assigned to their first-choice school increases with their score. Higher-achieving students typically attend high-performing schools, whereas low-achieving students typically attend the lowest-performing schools.

According to the school assignment requirements, there exist three categories of schools I considered as possible sites for this study. The first category consists of assisted (denominational) schools, which are usually the higher-achieving schools on the MOE's list and students' first choice. These schools usually receive students in the top 90th percentile in the SEA examination (Campbell, 1997; Jackson, 2012; London, 1989). All the schools in this category offer secondary school instruction for forms one through six (Ages 11 to 18). Several researchers

[11] The Secondary Entrance Assessment is a universal system of education, which prepares all elementary school students for entry into a secondary school. Students are assigned based on continuous assessment and their overall performance in the terminal SEA examination.

have classified these schools as "elite" schools based on their ranking and students' performance at a national level (e.g., Campbell, 1997; De Lisle, Smith & Jules, 2005; Jackson, 2010; London; 1989).

The second category consists of government schools. These schools receive students ranging from the lowest achievement to the 90th percentile in SEA. Approximately 95 % of these schools offer secondary schooling for forms one through five (Ages 11 to 16). The remaining 5% offer schooling for forms one through six (Ages 11 to 18). Schools within this 5% are ranked higher than those in the other 95%.

The third category consists of government schools, which are ranked among the higher-achieving schools and are thus also considered as "government elite schools" (Jackson, 2012). These schools unlike other government schools receive students in the top 90th percentile in the SEA examination and also offer schooling from forms one to six. There exists only two schools in this category in Trinidad and Tobago and I refer to them as government elite schools.

In order to obtain a representational sample of students and teachers in Trinidad and Tobago, I considered three schools for my study. My selections were guided by the three aforementioned categories of ranked schools and my goal of obtaining students of varying ability levels for my study. I selected a school from each category ensuring that the schools were in close proximity. The reason for this was to facilitate the possibility of data collection at the three school sites on the same days, during normal school working hours[12]. After narrowing my options, I sent a recruitment email to the principals of the three schools. After receiving approval from the principals, I contacted one teacher at each of the respective schools and they gave voluntary consent to participate in the study.

[12] In a centralized curriculum, the same topics will be possibly taught around the same day or weekly schedule.

At each school site, I observed only one grade or class level of students. This was dependent on the teacher who agreed to participate from each selected school site. The selection of the class level was based on the available classes taught by the teacher participating in my observational study. The age differences among the three levels did not impact my sample and observational data because all students were taught the same content aligned with the centralized curriculum and textbooks in Trinidad and Tobago. The uniformity of the content in the classes I observed ensured that the students' age difference did not affect my sample.

Additionally I considered classes that started the CSEC Mathematics syllabus in preparation for the CSEC Mathematics examination. For example, the respective levels were form five (ages 15 to 16) at School A, form four (ages 14 to 15) at School B, and form three (ages 13 to 14) at school C. The teacher at each school was assigned one of these classes for the 2015 – 2016 academic year. In Trinidad and Tobago, schools usually begin preparation for the CSEC Mathematics exam in form four. However as in the case of the government elite school (School C), these preparations begin in form three. Therefore, I selected the form three-class level from this school. Table 3.1 shows a summary of the school sites for my selection of teachers for this study.

As shown in Table 3.1, the three schools I selected were representative of the categories of schools I previously described. School A was a government assisted single-sex school with a form five class. School B was a government secondary school with a form four class as the site of my study. School C was a government elite school and the class level was form three.

Table 3.1

Demographics of School Sites

School	School Category	School Type	Class Level
A	Assisted Denominational	Single Sex	Form 5
B	Government	Co-ed	Form 4
C	Government Elite	Co-ed	Form 3

NOTE: Form 3 = Ages 13 to 14; Form 4 = Ages 14 to 15; Form 5 = Ages 15 to 16

Participants

The participants for this study were three secondary school teachers (adult male or female who are 18 years old and older) and their students (male or female who are 13 to 16 years old). To recruit the teachers, I contacted the principal of possible school sites through their school board or the Ministry of Education in Trinidad and Tobago. On receiving the principal's consent for participation, I contacted possible teachers with a recruitment letter via email. Once a teacher provided their voluntary consent, I distributed letters of student assent and parental consent students and parents before observations. Table 3.2 shows the demographic of the teachers who participated in this study. I present each teacher's pseudonym, school, class level, qualifications, and years of experience.

The teachers varied in their experience and teaching qualifications. The teachers had years of teaching experience ranging from 5 to 30 years. In Trinidad and Tobago, teachers are only required to have a major or minor in their subject area to be eligible to teach at the secondary school level. Teachers are not required to possess a bachelor of education degree. Additional there is no mandatory requirement for teacher certification. However, teachers can choose to pursue a graduate diploma in education for promotion to senior teacher or head of department for their subject area at their school. All three teachers majored in mathematics at

college. Two of the teachers earned their degrees in education, with only one teacher possessing the graduate diploma in education. Additionally, my final selection of teachers was based on recommendations from the principal and the head of the Mathematics department at each school. These recommendations were based on the teachers' overall expertise and leadership within their Mathematics departments. The principals considered teachers who seemed to be an integral support and guide for their fellow colleagues in developing lessons and instructional ideas. With regard to their expertise, the recommendations were based on teachers' who consistently demonstrated innovative approaches to their instruction and willing shared their expertise with their fellow mathematics teachers.

Table 3.2

Demographics of Teachers Participating in Study

Pseudonym	School	Class Level	Years of Teaching Experience
Ms. V	School A	Form 5	More than 10 years
Ms. M	School B	Form 4	Less than 10 years
Mr. P	School C	Form 3	More than 25 years

Each teacher taught one of the CSEC preparatory classes at their respective school for the 2015 to 2016 Academic year. These classes varied in the level of students. For example, at School A, the class level was Form 5 (ages 15 to 16). These students were in the final year of preparation for the CSEC examination in June 2016. At School B, the class level was Form 4 (ages 14 to 15), where the students were expected to write the CSEC examination in June 2017. At School C, the class level was Form 3 (ages 13 to 14); these students were in their first year of

preparation for the CSEC examination in June 2018. I selected these levels based on the availability of classes assigned to the teacher at the respective schools. The variation of class level also helped in obtaining a representative sample of students who usually prepare for the CSEC mathematics examination. I also considered whether the teacher began teaching content specific to the preparation of students for the CSEC mathematics examination. I also selected classes where the teacher did not previously cover or reviewed the content on triangle congruency in the CSEC mathematics syllabus.

Units of Analysis

The unit of analysis for the case studies was the instructional practices surrounding the teachers' creation and implementation of opportunities for reasoning and proof. In particular, my analysis focused on four aspects of the teachers' instructional practices. The first aspect was the sociomathematical norms the teacher established during the lesson. This was important because I was interested in how the implications of a social constructivist perspective to teaching and learning in the reform policies impact the instruction of reasoning and proof. The second aspect of focus in my analysis was the specific classroom mathematical practices related to teaching the selected mathematical topic. These practices referred to the accepted mathematical customs teachers allowed or encouraged their students to use when proving congruency of triangles. For example, teachers reminded students to use the congruency theorem such as *Side-Angle-Side* (SAS)[13] to prove two triangles are congruent or to guide their reasoning when observing patterns among congruent triangles. My analysis of the classroom practices was important for understanding how the teachers guided students to proving within a social constructivists

[13] The SAS theorem states that two triangles will be congruent if two sides and the included angle of one triangle are congruent to the corresponding two sides and included angle of the other triangle (SAS).

perspective. The third area of focus was the teachers' use of textbooks and CSEC examination preparatory materials while teaching the lesson. My intention was to analyze how and when these materials were used during the lesson. This area of focus addresses my third research question.

The fourth area of focus concerned my observation of any characteristics of reform-based teaching as defined by Hufferd-Ackles et al. (2004). This involved the examination of characteristics of a math-talk learning community during the teaching of reasoning and proof. I observed and coded the four characteristics according to the descriptions I adapted from Hufferd-Ackles et al. (2004) theoretical framework. The four characteristics were (a) type of questioning, (b) explanation of mathematical thinking, (c) sources of mathematical ideas, and (d) responsibility for learning reasoning and proof.

Data Sources

To obtain the case study reports for each teacher, I used (a) classroom observations (b) pre-observation interviews, and (c) post-observation interviews. I videotaped the classroom episodes, using two cameras and three audiotape recorders. One camera was placed at the front of the room to focus on the students' interaction with the teacher and the other camera was placed at the back of the room. This allowed a full view of the entire class and the ability to capture any work written on the chalkboard or whiteboard. I also used three audio recorders to capture the dialogue during classroom discussions. I placed one audio recorder in the front of the room to record the teacher's voice during the lesson. I placed the other two recorders when needed, among students to record their dialogues during whole class discussions or group work.

Classroom observations. I conducted classroom observations over the duration of the teaching of entire proof-based Geometry lessons on *Congruent Triangles*. The number of

classroom visits was subjected to the number of class sessions required to complete the topic. Table 3.3 shows the duration of the lessons at each school and the location of the lesson. Each observation lasted the duration of a scheduled mathematics period (45 minutes to 1 hour). The observations took place in classrooms where the lesson was taught. At School A and School C, the lessons were taught in the school's respective mathematics room. At these two schools there were three observations totaling 150 minutes of observation at each school. At School B, I observed the two lessons in the homeroom of the respective Form 4 class.

Table 3.3

Duration of Observed Lessons per Teacher

Teacher	Duration (minutes)			Total (minutes)
	Lesson 1	Lesson 2	Lesson 3	
Ms. V	60	60	30	150
Ms. M	35	70		105
Mr. P	60	30	60	150

During classroom observations, I constructed field notes guided by an observation protocol (see Appendix A). The observation protocol consisted of questions designed to capture (a) the sociomathematical norms teachers establish (b) the classroom mathematical practices evident during the lesson, and (c) the teachers' use of the textbook and assessment materials during the lesson. After each observation, I used my field notes and video recordings to write concise narrative descriptions of what happened during the observation by answering questions from the protocol. I also obtained both student assent and parental consent to collect samples of students' work during the days when a lesson was observed. The purpose of this was to determine the type of mathematical reasoning students did during the lesson.

Pre and post-observation interviews. I conducted audiotaped semi-structured interviews with participants before and after teaching a lesson. The interviews lasted 30- 45 minutes and were conducted during a free period or after school. I conducted the pre-observation interview (see Appendix B) to gain insight into the planning of the lesson, its objectives, and expected outcomes regarding students' learning of reasoning and proof. I intended to gather evidence of teachers' pedagogical decisions from the pre-observation responses. There are four components of teachers' pedagogical decisions. These are (a) the choice of mathematical tasks teachers implement (b) their processes of creating opportunities for reasoning and proof, (c) teachers' instructional strategies (e.g., questioning, revoicing, direct instruction or cooperative learning), and (d) teachers expectations for students (Martin et al., 2005). In column 2 of Table 3.4, I list questions, which I anticipated would provide data about specific components of teachers' pedagogical decisions in column 1. Responses to these pre-observation questions helped enrich the data I collected during the observations. For example, when I asked the following questions "What are your goals for the lesson?" or "What strategies are you implementing to achieve these goals?" teachers provided data about strategies they intended to use while teaching reasoning and proof.

Table 3.4

Pre-observation Data Sources for Identifying Teachers' Pedagogical Decisions

Component of Teachers' Pedagogical Decisions	Pre-Observation Interview Questions	
Choice of mathematical tasks.	Q2.	What topics do you plan to teach today?
Processes of creating opportunities for reasoning and proof	Q3.	What are your goals for the lesson?
	Q5.	Can you tell me about any activities that engage your students in doing reasoning and proof?

	Q6.	Which textbook do you prefer to use when teaching? Can you tell me more about why you prefer it to the other available options?
	Q7.	Did you use other resources or materials in preparing this lesson? (If yes, can you describe the materials and how you used them)?
Instructional strategies teachers intend to use	Q4.	What strategies are you implementing to achieve these goals?
Teachers' expectations of students	Q1.	Can you tell me something about the class you are teaching today? How are the students?

Additionally, the responses revealed whether developing students' competencies of reasoning and proof were explicit goals of the lesson. Responses to these questions also enriched observational data by identifying possible changes in the teachers' strategies during the lesson. This was important because it indicated the possible influences of the classroom microculture or curricula materials on teachers' pedagogical decisions as shown in my theoretical framework. I transcribed and coded the interviews to obtain any emerging relevant themes for further analysis.

I conducted the post-observation interviews (see Appendix C) after an observation, which contained classroom episodes pertinent to the goals of the research and in which affirmation, and clarification was needed from the teachers' point of view. The interviews took place in a classroom, or a site on the school premises that was conducive to interviewing and convenient to the teacher. Due to the cases where a teacher was not available immediately after an observation, I conducted the interview in the closest available time period after the class observation. The goal of the interviews was to obtain the teachers' perspective on classroom practices, as well as to learn more about their decisions during the lesson. The interviews also supported observational data by clarifying and informing reasons for the teacher establishing any

sociomathematical norms or classroom mathematical practices during the lesson. For example, when I asked the question, "What do you think went well in the lesson today?" Teachers' responses provided insight into why they established certain sociomathematical norms or classroom mathematical practices to achieve the lesson's goals. To further understand the implications of the reform's instructional policy and actual teaching practices, I asked the following general question, "In your opinion, how can teachers develop their students' ability to do reasoning and proof in mathematics?" Teachers' responses to this question highlighted any issues surrounding the teaching of reasoning and proof in Trinidad and Tobago.

Methods of Analysis

Data analysis primarily involved coding classroom observation field notes and video recordings for the sociomathematical norms, classroom mathematical practices, teachers' pedagogical decisions, and teachers' use of curriculum materials. I coded the observational data for each of the aforementioned areas to identify patterns that addressed my research questions. After identifying patterns, I consulted both the pre- and post-observation interviews responses to provide insight into inferences I made to determine the patterns.

Analysis and Coding of Video-Recordings

I conducted the analysis of the data in three stages. In Figure 3.3 I present a flow chart of the stages of my coding and data analysis. The analysis and coding of each stage either guided the coding in the next stage or independently contributed to my findings. In stage 1, I initially coded the field-note records of observations by changes in the activity structure (e.g., Introductory warm up (IWU), homework review (HW), or whole class discussions (WD)) or participant structure (e.g., Teacher-directed (TD), individual seat work (ISW), or group work

(GW)). This first step gave a general idea of the flow of the lesson before viewing the recording. I used these codes to construct a timeline of events of the videos.

In stage 2, I constructed a sequenced timeline of events for each video recording by identifying units of analysis called *instances of reasoning and proof*. I distinguished these from the other kinds of classroom events. The instances of reasoning and proof were specific times during the lesson where students engaged in pattern identification, conjecturing, or providing non-proof arguments, and constructing proofs. I further subdivided the instances of reasoning and proof into (a) non-proof justification events or (b) formal proof event. A non-proof justification event referred to instances when students developed rationales without using formal logical structures to justify a result. For example, this would include using diagrams or non-deductive arguments to explain one's reasoning. A formal proof event concerned the sociomathematical norm the teacher established that defined what counts as a valid formal proof in their classroom. I hypothesized that these may be unique to the cultural classroom practices of Trinidad and Tobago. I coded the other classroom events according to the initial activity structure codes from stage 1 of my analysis.

Analysis and Coding of Sociomathematical Norms

In stage 3, I analyzed and coded the video segments containing instances of reasoning and proof. I focused firstly on occurrences of sociomathematical norms, then the classroom mathematical practices, and finally the use of curriculum materials. I specifically focused on teacher and students discussions to understand how and when the teacher established these norms. By analyzing classroom discussions I identified moments where a teacher established a sociomathematical norm or moments when teachers and students negotiated these norms according to the didactical contract (Brousseau, 1997). I transcribed and coded the dialogues

where teachers specifically stated sociomathematical norms. I used emerging codes as labels for the sociomathematical norms that were evident in teachers and students' talk. For example, if a teacher explicitly establishes what is counted as an acceptable proof, or what is counted as an acceptable explanation, I assigned suitable codes to these sociomathematical norms. I also assigned codes such as telling (T) or negotiation with students (NS) to indicate how the sociomathematical norm was established. I also recorded possible purposes of the sociomathematical norm. By this I mean, whether it acts as a precursor to identifying patterns, conjecturing, or constructing non-proof or proof arguments. This was important for understanding when sociomathematical norms became instrumental in helping students engage in reasoning and proof. After coding and analyzing the video clips with instances of reasoning and proof, I analyzed and coded the other segments for possible occurrences of sociomathematical norms.

Analysis and Coding of Classroom Mathematical Practices

To identify the classroom mathematical practices associated with the selected topic, I coded the video clips with the instances for reasoning and proof, and the other video clips with other class activities. I paid attention to any moments the teacher or students established a classroom mathematical practice. For example, the teacher or student may remind the class to use a specific theorem (e.g., *Side Angle Side, SAS*) to solve a problem on congruent triangles. In this case, I assigned an appropriate code to this mathematical practice. I also noted the frequency of use and when the classroom practice was introduced during the lesson. I assigned suitable codes for the classroom mathematical practices evident during the lesson.

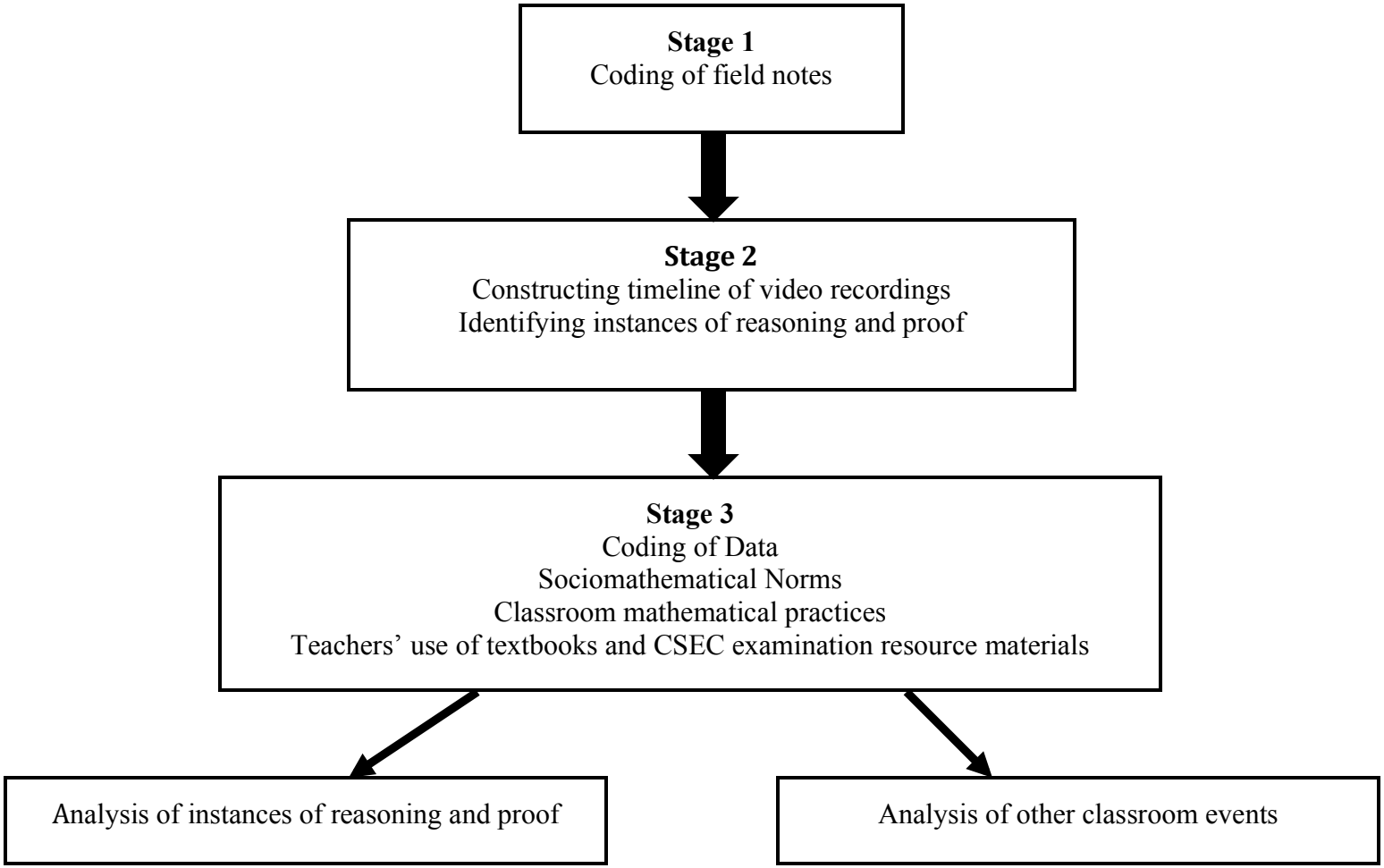


Figure 3.3: Stages of data analysis.

Analysis and Coding of the Use of Curriculum Materials

I used both teachers' responses from the observation interviews and video footage to code and analyze their use of curricular materials. I assigned the codes to describe how and when teachers used textbooks and CSEC examination resources during their instructions. My proposed list of possible categories for coding teachers' use of these materials were:

- Teaching content
- Referencing definitions, axioms, and theorems
- Assigned homework
- Assigned in-class practice exercises
- Demonstration of the construction of proofs
- Assessment of students reasoning and proof abilities

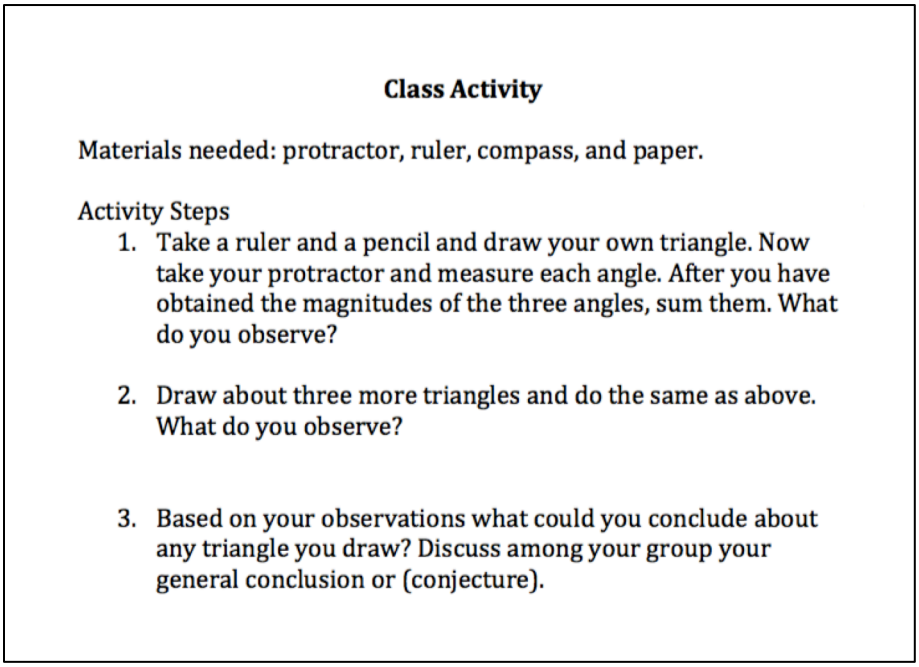
I used the codes to help identify the occurrences of teachers' use of textbooks and CSEC materials in the video clips and interview responses. I also incorporated any emerging patterns of usage evident in the video data.

Example of Coding and Analysis

In the following vignette, I present an example of an instance of reasoning and proof from one of the classes of Ms. M at School B. The vignette presents the investigation of the interior sum of a triangle theorem. Following the descriptions of the teaching episode, I present an example of my coding and analysis of the data for the (a) opportunities for reasoning and proof, (b) sociomathematical norms, and (c) classroom practices.

In the following excerpt, Ms. M was teaching the interior angle sum of a triangle theorem. After the initial warm up exercises and the correction of homework from the previous day, the following dialogue took place.

Ms. M stands at the front of the room and addresses the students "Okay I want you take out your protractor and pencil and in your groups, do the activity on the board." Students take out the protractor, rulers, pencil, and writing material and begin to work on the activity in groups of four. Figure 3.4 shows the activity as indicated on the board. Ms. M walks around the room for about 8 minutes talking to the groups and observing their discussions. After visiting three of the groups she says, "Remember I want you to discuss with your members what you observe. You have to come up with a conjecture and you have to explain with reasons to support your answer." The students listen to the announcement and then continue working on the activity.



Class Activity

Materials needed: protractor, ruler, compass, and paper.

Activity Steps

1. Take a ruler and a pencil and draw your own triangle. Now take your protractor and measure each angle. After you have obtained the magnitudes of the three angles, sum them. What do you observe?

2. Draw about three more triangles and do the same as above. What do you observe?

3. Based on your observations what could you conclude about any triangle you draw? Discuss among your group your general conclusion or (conjecture).

Figure 3.4: Class activity for opportunity of reasoning and proof. Adapted from Lesson 1 of Ms. M at School B.

After five additional minutes, Ms. M signaled to the students to stop writing and asks for a group to volunteer their solution. Members of group two raised their hands quickly before the other five groups. Ms. M acknowledged their hands and asks for a representative to report their solution. Oliver stood and walks to the front of the class to explain his groups' solution. Oliver stated that after summing the angles in their first triangle they found that the sum of the angles was 180^0. Oliver wrote on the board while narrating "$35.8° + 110° + 34.2° = 180°$." Ms. M then asked, "What else did you do? How about the next step?" Oliver then proceeded to explain that by drawing three additional triangles and finding the sum of the interior angles he noticed that in all cases the sum was 180^0. Ms. M nodded and then said, "Okay continue." Oliver then said, "Well based on the pattern I got here and also each member of my group got the same pattern we decided to come up with a conclusion for any triangle." Ms. M said, "Good so what is it?" Oliver responds while writing on the board, "The sum of interior angles of any triangle is 180º." Ms. M re-voices Oliver's conjecture and asked: "Did anyone get anything different?" None of the students raise their hands. Ms. M then asked, "Did everyone make a similar conjecture? Raise your hands if you did." All the students raised their hands. Ms. M says, "Good job, let us now prove this result."

Analysis and Coding of Vignette

In the vignette, when Oliver described his solution and conjecture it represented an example of an instance of reasoning and proof. I considered this occurrence as an opportunity for students to engage in pattern identification and conjecturing. As shown in Figure 3.4 students were required to construct any triangle using a ruler and pencil. On completion of this task, the students needed to measure the interior angles of the triangle they drew and find the sum of the interior angles. The phrase "What do you observe?" seemed to suggest that the students were

expected to identify a pattern about the sum of interior angles of the triangles they drew. In the third step, students were also asked to draw a conclusion or conjecture about the sum of interior angles of a triangle. Because students did not write a proof of the interior angles sum theorem, I did not code this reasoning and proof instance as a formal proof event. Table 3.5 summarizes my analysis of this episode.

Table 3.5

An Example of Analysis and Coding of a Teaching Episode

Activity Codes	Teacher Actions	Student Actions	Coding	
			Opportunity for Reasoning and Proof	Sociomathematical Norm
IWU, TD	Ms. M introduces students to the activity. Ms. M stands at the front of the room and addresses the students "Okay I want you take out your protractor and pencil and in your groups, do the activity on the board."	Students take out the protractor, rulers, pencil and writing material and begin to work on the activity in groups of four.		
GW	Ms. M walks around the room talking to the groups and observing their discussions. After visiting three of the groups she says, "Remember I want you to discuss with your members what you observe. You have to come up with a conjecture and you have to explain with reasons to support your answer."	The students listen to the announcement and then continue working on the activity.		VS: A valid answer must have supporting reasons: Precursor for PI and MC

GW	Ms. M continues walking around the room visiting the groups and listening to their discussions	Students continue working among their groups asking each other question. In some cases student are not aware Ms. M is standing listening to their discussions. The students in these groups are actively discussing their solution.		
TD	Ms. M signals to the students to stop writing and asks for a group to volunteer their solution.	Members of group two raise their hands quickly before the other five groups.		
IRP	Ms. M. acknowledges their hands and asks for a representative to report their solution.	Ornaldo stands and walks to the front of the class to explain his groups' solution. Ornaldo states that after summing the angles in their first triangle they found that the sum of the angles was 180^0. Ornaldo writes on the board while narrating "$35.8° + 110° + 34.2° = 180°$"	Instance of Reasoning and Proof: **PI**	
IRP	Ms. M then asked, "What else did you do? How about the next step?"	Ornaldo then proceeds to explain that by drawing three additional triangles and finding the sum of the interior angles he noticed	Instance of Reasoning and Proof: **PI**	

		that in all cases their sum was 180^0.		
IRP	Ms. M then prompts Ornaldo by asking, "What else did you do? How about the next step?"	Ornaldo then proceeds to explain that by drawing three additional triangles and finding the sum of the interior angles he noticed that in all cases their sum was 180^0.	Instance of Reasoning and Proof: **PI**	
IRP	Ms. M nodes her head and then says, "Okay continue".	Ornaldo then says, "Well based on the pattern I got here and also each member of my group got the same pattern we decided to come up with a conclusion for any triangle."	Instance of Reasoning and Proof: **MC**	
IRP	Ms. M says "Good so what is it?"	Ornaldo responds while writing on the board, "The sum of interior angles of any triangle is 180^0."	Instance of Reasoning and Proof: **MC**	

NOTE: IWU= Introductory warm up; TD= Teacher-directed discussion; GW= Group work; IRP= Instance of reasoning and proof; PI= Pattern identification; MC= Making conjectures; VS= Sociomathematical norm of what is a valid answer.

As shown in Table 3.5, I used the following codes to identify the general class activities in column 2. The codes were, (a) Introductory Warm Up (IWU), (b) Group Work (GW), (c) Teacher-directed discussion (TD), and (d) Instances of Reasoning and Proof. Overall there were three instances of reasoning and proof (See Table 3.5). In these instances, students were engaged in pattern identification (PI) and making conjectures (MC). Both of these processes were requirements of the given activity. I considered the teacher and students talk during Oliver's presentation to gain evidence of the respective reasoning and proof activity. For example, when Oliver stated that "Well based on the pattern I got here and also each member of my group got the same pattern we decided to come up with a conclusion for any triangle," it seemed to suggest his groups' identification of the pattern that would lead to their conclusion or conjecture. In this case, I coded this instance as pattern identification (PI) and making a conjecture (MC). Oliver went further to state that his groups conjecture was "The sum of interior angles of any triangle is 180º." The previous mathematical reasoning among the group supported the making of this conjecture where Oliver and his other group members identified a pattern about the sum of interior angles. Additionally Oliver suggested that the observed pattern supported their conjecture. Oliver's explanation seemed to adhere to the teacher's sociomathematical norm introduced earlier in the vignette

With regard to the sociomathematical norms, the teacher established one of these in her introductory explanation of the activity. The teacher articulated that "You have to come up with a conjecture and you have to explain with reasons to support your answer." This quote seemed to suggest the establishment of the sociomathematical norm of what constitutes as a valid solution to the given activity. According to the teacher, a valid answer would have supporting reasons explaining the validity of the result. I coded this sociomathematical norm as the definition of a

valid solution (VS). Because the teacher's presented this norm to clarify her expectation of students work in identifying patterns and making conjectures, I coded this norm as a precursor for these two reasoning and proof processes.

Reliability Coding

I conducted reliability coding with a fellow researcher. We met and discussed my coding stages and codes as outlined in Figure 3.3. After our discussions, we randomly selected one lesson from each teacher. My colleague viewed the video recording of the lesson and coded according to the stages I described earlier and shown in Figure 3.3. After our coding of instances of reasoning and proof we compared or codes and we obtained 90% reliability. We discussed any discrepancies and most were associated with the meaning of some dialect phrases during some of the recorded teaching episodes. After we reached an understanding of the cultural meaning of these phrases we continued further coding and comparisons of the selected video recordings for each teacher. To conduct reliability coding of the sociomathematical norms we identified dialogues, which exhibited these areas and coded the dialogue transcripts for these segments according to the codes such as telling (T) or negotiation with students (NS) to indicate how the sociomathematical norm was established during the lesson. With regard to this coding we obtained a 92% agreement.

Results

In the following sections, I provide descriptions of the teachers' instruction. I describe one entire lesson, which demonstrates typical opportunities for reasoning and proof during my observations for each teacher. I selected a lesson based on the evidence supporting the teachers' classroom practices, sociomathematical norms, and their use of curriculum materials. The objective of these descriptions is to exemplify the type of teaching strategies and pedagogical

decisions of the various teachers during the teaching of reasoning and proof. Throughout the descriptions I will provide contextual evidence supported by excerpts or explanations from the pre- and post-observation interviews. Before presenting the lesson description, I provide a profile of each teacher, which describes their leadership roles and expertise within their school's mathematics department. As I explained earlier these unique characteristics were influencial in each school principal's recommendation of teachers for my final selection of suitable participants for this study. Overall the descriptions of teachers' profiles and their instruction provide a descriptive case study of each teacher's instruction. Following these descriptions, I will present my analysis of the observational data according to the classroom microculture, teachers' use of curriculum materials, and evidence supporting reform-based teaching.

Profile of Ms. V

Ms. V indicated that her teaching of mathematics is motivated by getting her students to appreciate their mathematical experiences. Ms. V asserted during her pre and post-observations interviews that she sees her preparation of each lesson as an opportunity to be innovative in using students' interests to create opportunities for engaging them and connecting their mathematical experiences with real-world activities. When I asked the question " How do you come up with ideas for your lessons?" during the post-observation interview of day two of her teaching, Ms. V stated: "I usually like to use a lot of other resources from the internet like the National Council of Teachers of Mathematics (NCTM) Illuminations lesson ideas to get really innovative ways to present my lesson and I tailor them if needed to meet students' interests." Based on my observations, Ms. V utilized several of these online resources to create interactive activities for her students.

Other teachers at her school considered Ms.V an expert in using technological tools during instruction. By this I mean that she incorporated various types of technologies in her teaching such as short videos, interactive hands-on activities on the smart board, and power point presentations. During my time at her school, several of her colleagues sought her assistance during her free-periods and lunch break for advice. Ms. V's colleagues asked her assistance for incorporating the aforementioned technological tools in their respective mathematics lessons. Based on my observations it seemed that Ms. V was a valuable resource at her school because of her expertise and willingness to share ideas and her time to assist other colleagues. Additionally Ms. V was highly respected for the quality of her supplementary worksheets she prepared for each topic in the schools' mathematics curriculum. Ms. V indicated in response to my question: Which textbooks or resources do you usually use to plan your lesson?"

> Ms. V: Well, I use the textbook only for preparing the worksheets I give students in class. I would also use past CSEC examination questions and also some of my own questions for students to practice in the worksheets.

Ms. V's quote indicates her use of the textbook for her teaching. Ms V also asserted her use of CSEC examination materials to prepare the worksheets for students' assigned work. Ms. V's investment in creating worksheets for her students demonstrates her sense of innovation and her goal to promote students' learning opportunities. Overall Ms. V seemed to be well respected and valued by her colleagues for her expertise and resourcefulness in creating mathematics lessons, applying technology, and developing worksheets.

Vignette of Ms. V: Solving Joe's Problem

In this section I present a description of one of the lessons Ms. V taught on congruency. This was the second of three lessons covering Congruent Triangles. The session lasted for 70 minutes. All 35 students were present. In this lesson students are finding the minimum conditions

for congruency of a pair of triangles. Ms. V uses a problem-based lesson, which requires students to solve a problem about building triangular roofing trusses for a construction worker named Joe. The solution of the problem entails finding the four postulates of congruency of triangles.

Opening Vignette

The bells rings indicating the start of lesson two of Ms. V's instruction on congruency. Ms. V begins the lesson by explaining to the students the format of the lesson, which would include group work, and a problem solving activity entitled "Slow Joe." Ms. V also reminds her students of the rules for today's session. She talks about the requirement of always providing explanations for your solutions. She also reminds the students that they must give their undivided attention to a fellow student or group when they are reporting or giving their solutions. The students listen intently to her announcements and take out their notebooks. Ms. V then goes to the smart board located to the right wall of the classroom and puts up the first slide with the words "Pre-Knowledge" written in bold letters across the screen.

Investigating Congruency

As students look towards the smart board, Ms. V paces in front the board talking about what was taught in the previous lesson. During the first lesson, Ms. V guided students to create their own definition of congruency of triangles. Ms. V solicits a volunteer to state their definition of congruency from the previous lesson. Ms. V asked the question, "Could anyone share his or her definition of congruency from our last session?" Monique volunteers and states her definition, "congruent triangles have corresponding sides and angles." Ms. V smiles and says "Good job. Now let us look at the definition on the smart board."

The students adjust their seats and look up at the smart board to the right of the room. Ms. V shows the slide with her definition of congruent triangles. The slide states, "Two triangles are

congruent when the six components of one triangle are equal to the six corresponding components of the other triangle." Ms. V explains that the six components should include the three sides and three interior angles. Ms. V emphasizes the word corresponding in her definition. Students listen to the Ms. V's explanation about the six elements that must be checked to determine congruency.

Ms. V: Okay class as the definition states the six elements or components of one triangle must be equal to the corresponding components of the second triangle. Do you follow this?

Students: Yes Ms.

Ms. V: So what must be equal for a pair of triangles to be congruent? The…

Students: Corresponding components.

(The students read as Ms. V points to the bolded phrase in her definition on the white board).

Ms. V: Okay good. Let us move on.

Ms. V continues to talk about the corresponding sides and angles of a pair of triangles. The students write down the notes of the slides in their notebooks while listening to her explanation.

An Example of Congruent Triangles

Ms. V uses an example where there exist two triangles, *ABC* and *DEF* as shown in Figure 3.5. Ms. V explains that from the two triangles the sides $AB = DE$, $BC = EF$, and $CA = FD$. Ms. V explains that the appropriate markings on the respective sides indicate their corresponding lengths. For example, Ms. V indicates that the sides $\overline{AB}$ and $\overline{DE}$ both have a single stroke to indicate they are of the same length. As a result, the sides are corresponding. Ms. V explains that the corresponding angles can be located by using the appropriate markings for the angles. Students continue to write in their notebooks. Ms. V then talks about how the six corresponding elements (i.e., the three sides and three angles) must be checked so that one can conclude that the

triangles are congruent. Ms. V also talks about the correct labeling for corresponding elements. The students listen and take notes.

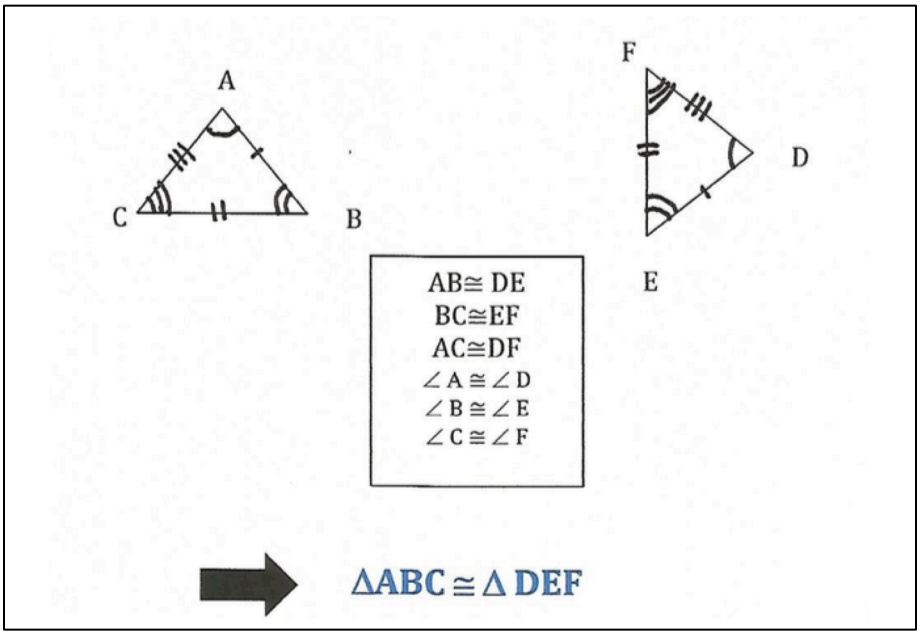


Figure 3.5: Example of the six corresponding elements for congruency of triangles.

Ms. V changes the slide and asks students the following question, "Do we always need all six elements to prove that two triangles are congruent?" Some of the students respond, "No" One student, Melanie, raises her hand and says, "Ms. it is no but I am not sure why?" Ms. V asserts Melanie's response and says, "Yes Melanie, you are correct. There is a minimum number of conditions one must check to determine congruency but the order of labeling the elements for these conditions is important." Students listen while writing in their notebooks. Some students nod, apparently in agreement with Ms. V's explanation. Ms. V proceeds to change the slide.

Identifying the Minimum Necessary Conditions for Congruency

Ms. V introduces the investigation of the necessary minimum conditions for triangle congruency. She shows the students on the smart board a real world example of congruency. The example shows the construction of a Ferris wheel as shown in Figure 3.6 below.

Figure 3.6: A real world example of congruent triangles in a Ferris wheel.

Ms. V asks the students to identify any congruent triangles in the Ferris wheel. Several students offer suggestions, all of these suggestions indicate one of the sections between the spokes in the wheel. Ms. V asks the students,

> Ms. V: Why do you think congruent triangles are needed? (*Jordan raises her hand, Ms. V gestures to her to begin speaking).*
>
> Jordan: Well if the triangles are not congruent then the wheel would not look circular and have the same radius for each section and this will not work properly.
>
> Ms. V: Okay, so Jordan is saying here that since the wheel is circular, each of the spokes here *(Ms. V points to the spoke in the wheel)* is a radius of the circle. Each of the spokes make up a side of one the triangles you are seeing here *(Ms. V points to one of the triangles in the wheel)* and we must have them being the same else the wheel would not work as Jordan stated.

Lisa raises her hand, Ms. V looks at her and she begins to speak

Lisa: Ms. would they be the same for each of the triangles if the triangles were congruent?

Ms. V: Yes that's correct Lisa. The congruent sides of the triangles that go from the center of the wheel to the circumference ensure the same radius and circular shape. Does everyone understand?

Students: *(In unison)* Yes Ms.

In the previous excerpt Ms. V used a real-world example of congruent triangles. Ms. V solicited explanations from her students so that they can discuss their understanding of the utilitarian characteristic of Geometry. Ms. V used the opportunity to help students such as Lisa to use the concept of congruence to understand the real-world example of a Ferris wheel.

Launching Joe's Problem

Ms. V changes the slide and launches the problem for the day entitled "Slow Joe." She shows students an example of congruency in home construction. She explains how congruent triangles are necessary for building roofing trusses. Figure 3 shows an example of a single triangular roofing truss. A roofing truss is a structure comprised of one or more triangular units, which are connected by metal plates at each joint. The external frame is a larger triangle with upper chords and lower chords as shown in Figure 3.7. The triangle shape gives the truss strength and stability. Trusses are often referred to as an open web system due to the web-like appearance caused by the triangular patterns. The design of the truss is for transferring weight of supported load of the roof to the joints, rather than to the chords or webs. Typically, roof trusses are used in residential and commercial construction roof framing.

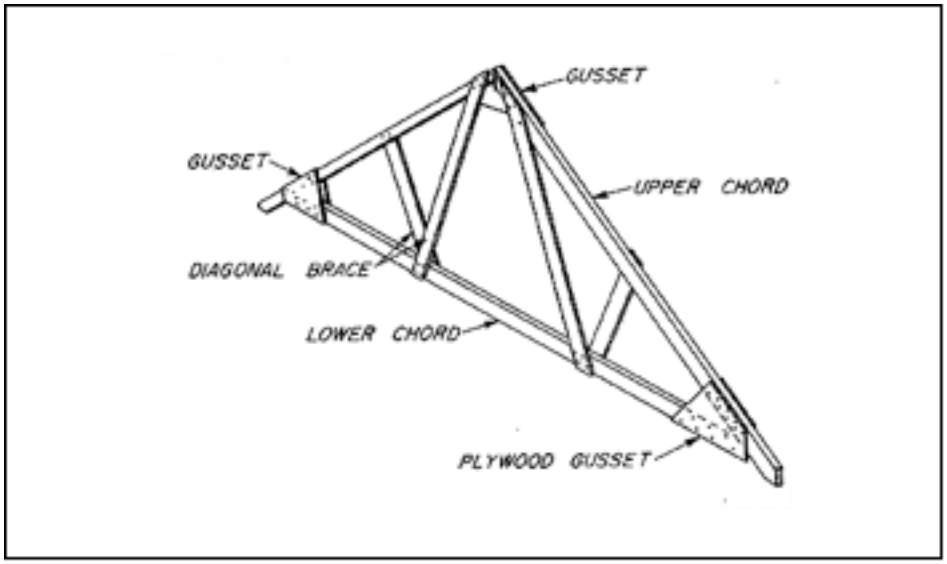


Figure 3.7: An example of a triangular roofing truss as shown in Ms. V's lesson.

Figure 3.8 shows a slide, Ms. V projected on the screen displaying the home construction example of congruent triangles. As shown in Figure 3.8 the triangular trusses are placed in the upper portion of a roof so as to maintain the structure that is triangular shaped. The trusses must be the same shape and size to hold the roof's structure as designed and keep the alignment of the roof. The upper and lower chords must be kept the same for each truss so as to maintain the structure.

Ms. V explains the problem of a construction worker named "Slow Joe." Joe is a roofing construction worker and has the task of making triangular trusses for the roof of a house his employer's construction company is building. Joe must ensure that each truss is the same. This means that they must all be congruent triangles. A roof may require 20 to 50 trusses. Joe usually checks all six corresponding elements of a triangle to construct the trusses. However Joe has been nicknamed "Slow Joe" by his fellow workers because he does not work fast enough when building the trusses.

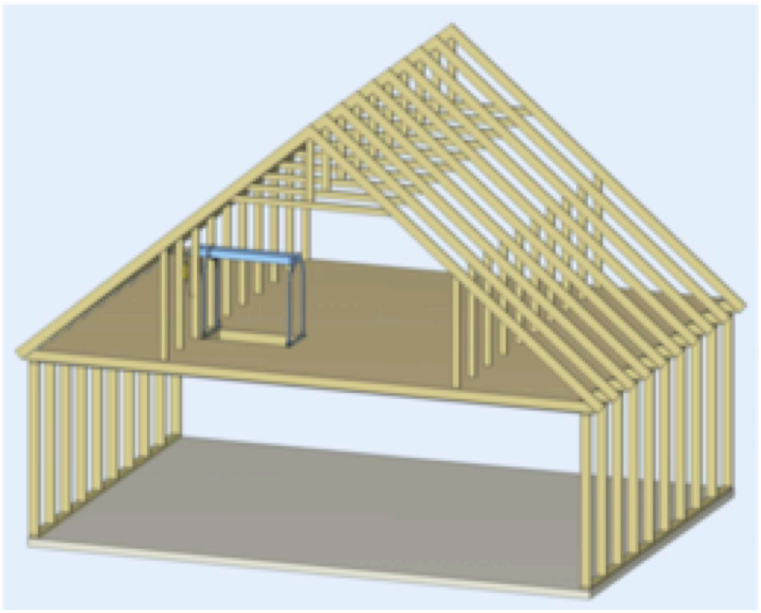


Figure 3.8: Ms. V's example of triangular roofing trusses for Slow Joe's problem.

As a result, Joe needs to work faster, thus his method of using all six elements of a triangle to make a congruent truss needs to be modified. Ms. V asks her students to come up with the minimum necessary conditions Joe could use to obtain congruent triangles for his trusses. Students read the info on the smart board and listen to Ms. V's instructions for solving the problem.

Student Activity: Helping Slow Joe

The activity stated that in groups of four, students must discuss and determine the least number of measurements Joe must use to produce congruent triangles for the trusses. Students were expected to write down their ideas and the measurements they think are needed to always produce congruent triangles. Ms. V asks students to divide into groups of four and discuss their solution to the proposed problem. Ms. V distributes examples of triangular trusses with the same measurements to each group (see Figure 3.9).

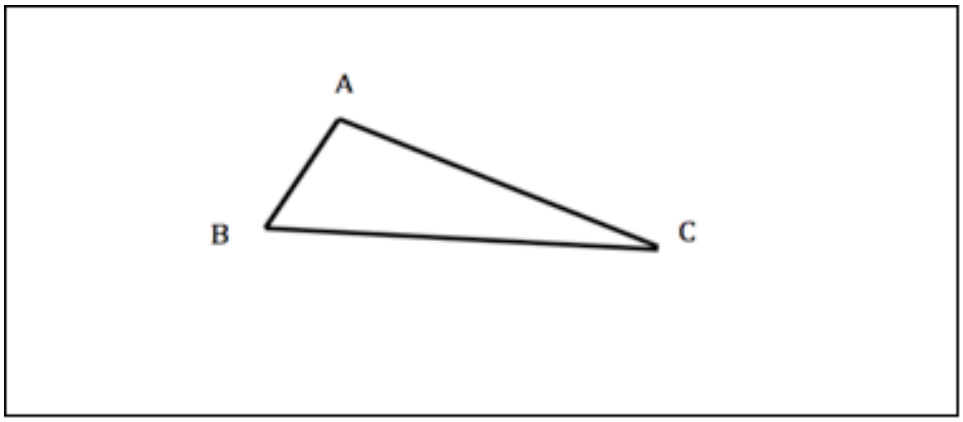


Figure 3.9: An example of the sample triangular truss for solving Joe's Problem.

The measurements for the sample roofing truss are $AB= 6.30$, $BC= 11.0$, $CA = 9.5$, $m\measuredangle A = 85^0$, $m\measuredangle B= 60^0$, and $m\measuredangle C= 35^0$. Students are required to cut out an identical triangular truss shown in Figure 3.9. Students are expected to find the minimum number of requirements they need to use to cut out the identical roofing truss. Students divide into groups and begin working on their activity. Ms. V asks her students to make a hypothesis about the necessary conditions for congruency. The conditions would include the required measurements they can use to cut out a triangle that is congruent to the one in Figure 3.9. Students must use less than six measurements. The aim of the activity is to obtain the minimum number of necessary measurements that will give a congruent pair of triangles.

After several groups begin making hypotheses, they begin to cut out triangles that are congruent to the one they were given. Students compare their triangles in their groups and indicate which minimum combination of measurements is necessary for always producing a pair of congruent triangles. Ms. V walks around the room observing the group's discussions. She answers questions to clarify the instructions. Ms. V continues to monitor the groups for about 20 minutes.

After observing the various groups' discussions for an additional five minutes, Ms. V makes an announcement. Ms. V reminds her students that they must use their hypotheses for the minimum necessary conditions for congruency to cut out a congruent triangle. Ms. V clarifies that the triangle must be congruent to the one initially given to the group (see Figure 3.9). The two triangles should represent a pair of congruent trusses for Joes' problem. The various groups continue to work on their assigned tasks for an additional five minutes. In some cases, the group members would ask Ms. V questions as they discuss their solutions. During her observations she listens to the discussions without providing any input.

Solutions to Joe's Problem

Ms. V walks to the front of the class and asks the groups to end their discussions and prepare themselves to present their solution. Students begin to pack up their materials and rearrange their seats while directing their attention to the front of the room. Ms. V is standing next to the white board waiting for students to end their discussions. When the classroom becomes quiet, Ms. V explains that for each group presenting, they must explain their solution. Ms. V indicates that she will label the hypotheses for the minimum conditions as conjectures. The class will need to test the proposed conjecture.

Tia's Conjecture

Ms. V asks for volunteers to present their proposed minimum condition for congruent trusses. Tia's group volunteers and she speaks on their behalf. Before Tia begins speaking, Ms. V reminds the class that they should pay attention and listen to a group's presentation. She also reminds the groups that when reporting their findings, they must provide the necessary clear explanations so as to assure that everyone understands. Tia begins to report and suggests that in their group they used the measurements *BC, AC,* and $m\measuredangle C$, giving their condition as Side-Side-

Angle (see Figure 3.10 below). As shown in Figure 3.10, Tia's group selected two sides and the enclosed angle indicated by (SSA). The solid lines indicate the measurements Tia's group used to obtain their congruent triangle.

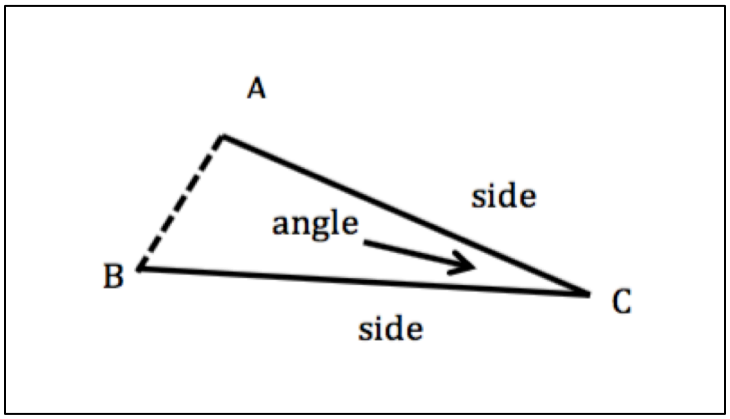


Figure 3.10: The measurements for Tia's necessary condition for congruency.

Ms. V writes on the white board, "Tia's Conjecture: Side-Side-Angle (SSA)." Ms. V then asks Tia the question, "Where should the angle be located?" Tia realizing that she forgot an important note about her suggestion responds saying, "I forgot, most importantly, the angle we used was between the two sides, so order is important" Ms. V repeats Tia's point by saying, "Okay class now as Tia stated the angle must be the one that is enclosed by the two sides so order is important." Ms. V solicits another group's suggestion. Vashti raises her hand.

Vashti's Conjecture

Ms. V acknowledges Vashti's hand and allows her group to present their suggestion. Ms. V writes on the board, "Vashti's Conjecture." Vashti reports that her group used the measurements *AB*, *BC*, and $m\angle B$, where the angle is between the sides (see Figure 3.11). The solid lines indicate the respective measurements that Vashti's group used to obtain their congruent triangle. Vashti repeats SSA, pointing at the notation Ms. V wrote next to Tia's conjecture on the board. Ms. V following Vashti's direction writes Side-Side-Angle (SSA) next

to “Vashti’s conjecture.” The students write down the second suggestion. Tia raises her hand to ask a question.

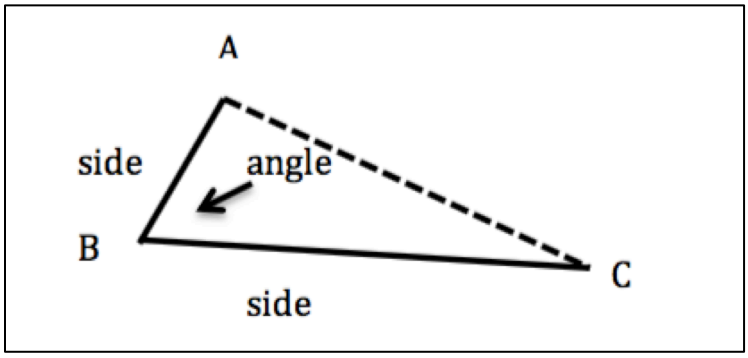


Figure 3.11: The measurements for Vashti’s necessary condition for congruency.

Fixing The Order Of The Measurements

Ms. V acknowledges Tia's hand and allows her to speak. Tia says, “Ms. because the order is important we should probably change the notation to Side Angle Side.” Ms. V agrees and discusses with the class that the order is crucial so we would change Tia’s and Vashti's conjecture to the same notation SSA to SAS. Figures 6 and 7 both show the correct ordering of the two sides and the enclosed angle as suggested by Tia.

Ms. V: Okay, we have a suggestion from Tia.

Students stop writing and listen to Ms. V.

Ms. V: Now, Tia is saying that we should change the notation from SSA to SAS because the order is important. What do you think?

Melissa raises her hand, Ms. M motions to her to speak.

Melissa: Ms. I think that is better for us to remember the order if you have SAS.

Tia: That’s the reason I think we should change it because the angle must be in the center of the two sides.

Ms. V: Does everyone agree?

Students: Yes Ms.

Nicki waves her left hand while writing in her notebook. Ms. V on seeing her hand says,

Ms. V: Okay, Nicki, do you have a question?

Nicki: Yes Ms.

Ms. V: Okay class let us listen to Nicky's question.

Nicki: Why should we change it?

Ms. V: Because it indicates the importance of having the angle enclosed by the two sides.

Nicki nods and says,

Nicki: Okay, I get it.

Ms. V: Do we agree?

Students: *(In unison)* Yes Ms.

Ms. V solicits another suggestion different from the previous two.

Cece's Conjecture

Cece raises her hand and proceeds to report for her group. Cece states that her group used the measurements, *BC,* $m\measuredangle B$, and $m\measuredangle C$ (see Figure 3.12). The solid lines indicate the measurements Cece's group used to obtain a congruent triangle. Cece emphasizes that the side is in the center of the two angles as shown in Figure 3.12 below. The solid line indicates the side and the arrows point to the two angles Cece's group used as their necessary condition for congruency.

Ms. V repeats Cece's suggestion while simultaneously writing on the board "Cece's Conjecture: Angle-Side-Angle (ASA)." Ms. V says, "Okay Cece's group used the side *BC* and the two angles that enclose this side. This gives us two angles and a side, which we will define as Angle-Side-Angle because the order is important here." The students listen and take notes.

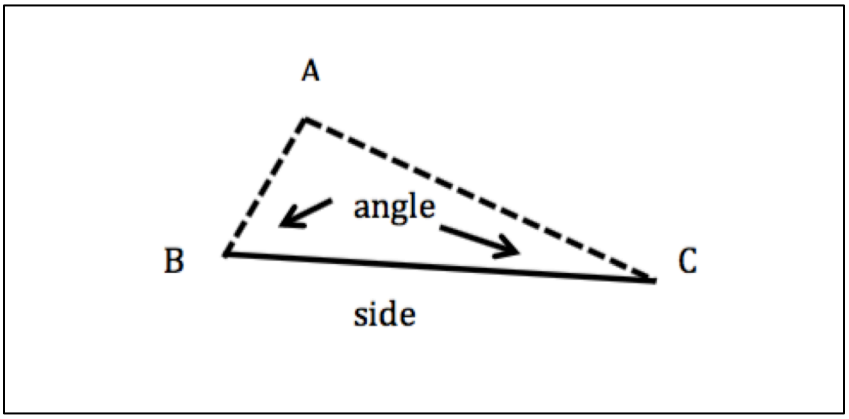


Figure 3.12: The measurements for Cece's necessary condition for congruency.

Testing of Conjectures

Ms. V asks the two groups who suggested SAS (i.e., Tia's and Vashti's) to compare the triangular trusses they made using their measurements. Ms. V walks to the center of the classroom and holds up the two triangles and compares them. Ms. V asks, "Are these two trusses or triangles congruent?" The students respond, "Yes" some students nodding in agreement. Ms. V asks, "Why?" Tiffany raises her hand and responds "Well Ms., they are congruent because they have the same size and shape and the components of the two sides and the enclosed angle match up." Ms. V repeats Tiffany's reasons by saying, "Tiffany says that the two triangles have the same size and shape meaning they are congruent because of SAS." Students listen while writing in their notebooks. Ms. V asks, " Does everyone agree with Tiffany's explanation." The students respond in unison "Yes Ms." and continue to write in their notebooks.

Ms. V walks to the front of the room and holds up the two triangles again so that all the students could see that they line up completely. She asks the class "anything else you noticed about these two?" Tia volunteers and explains, "Well our group used different sides and angles as Vashti's group. We used *BC, AC*, and $m \measuredangle C$ while they used *AB*, *BC*, and $m \measuredangle B$. Although

they are different we used the same condition of Side-Angle-Side and we got congruent triangles so I think we are on to something."

Ms. V seems to evaluate Tia's comment by nodding while smiling in agreement. Ms. V says "Well done and you are indeed on to something important." Ms. V tells the class that Tia and Vashti's conjecture is indeed correct and it ensures congruency even when they used different sides and angles. Ms. V explained later in her post-observation interview that her students agreed that a different solution could have the same final answer but employs a different method. Therefore, in this lesson, the two groups got the same condition SAS but used different measurements to make their hypotheses. As a result, students accepted the two answers as different. As students continue writing, Ms. V concludes by saying, "Okay class, based on our testing of the SAS condition with the different measurements, we see that it produces congruent trusses." Students listen, taking notes, and nodding their heads in agreement.

Ms. V then asks Cece's group to compare their triangular trusses with Tia or Vashti's trusses. She holds the three triangles up above her head and compares them. The triangles line up and she says, "What do we see here? Are they congruent? Why?" Cece raises her hand. Ms. V acknowledges her hand and gestures to her to speak. Cece says, "Yes they are congruent because they have the same size and shape and although we used two different conditions like SAS and ASA. We got two congruent trusses, so I believe we have another set of valid conditions for congruency." Ms. V nods her head in agreement and concludes by saying "Yes class we now have two valid necessary conditions for congruent triangles which are SAS and ASA. What do they mean class?" The class responds in unison "Side-Angle-Side" when Ms. V points to the acronym SAS. The students respond in unison "Angle Side Angle" when Ms. V points to the acronym on the white board next to Cece's name.

The Third Conjecture

Ms. V solicits another group to present a different set of conditions. One group's member, Gaby raises her hands and suggests Side-Side-Side (SSS). Gaby explains that her group's members used the sides *AB, BC,* and *AC*. As shown in Figure 3.13, Gaby's group used the three sides of the triangle as a necessary condition for their congruent triangle. Gaby also states that they checked their triangle with a nearby group who used ASA.

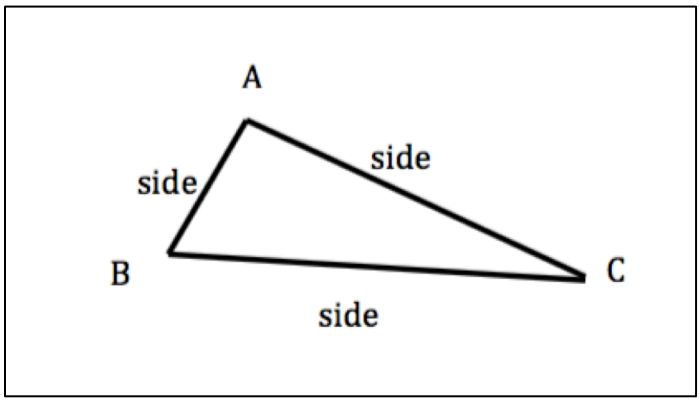


Figure 3.13: The measurements for Gaby's necessary condition for congruency.

Gaby holds up the two triangular cut outs and says "they are congruent so we know our suggestion is valid." Ms. V writes "Gaby's Conjecture" on the white board and writes the acronym Side-Side-Side (SSS). She nods during Gaby's explanation and eventually says "Well done" and then asks the question, "Does everyone follow Gaby's explanation?" The students say "Yes Ms." and also nod in agreement. Ms. V then reviews the three suggestions SAS, ASA, and SSS. The students listen and write notes.

The Counterexample of Angle-Angle-Angle (AAA)

Ms. V then asks the class "What about Angle-Angle-Angle (AAA)? Would this be a condition? Explain why or Why not?" Casey raises her hand and begins to speak after Ms. V points at her. Casey says, "Well, it is not possible because we showed this in a previous exercise

in our last class that AAA does not work." Ms. V repeats Casey's comments and proceeds to review the example from the last class. Ms. V reminds the students of the activity from the last lesson entitled "Finding your Twin." In this activity students were given triangles with different measurements. Each group of students was expected to measure the triangles using their protractor or ruler. The groups recorded the measurements.

Ms. V explains how each group had to use their recorded measurements to find the triangle that was congruent to the one they had been given. Ms. V talks about how Jordan's group had a triangle with the angles 80^0, 60^0, and 40^0. Mary's group had the same measurements but when the two groups compared their triangles, they were not the same size. Mary group's triangle was much larger than Jordan's; therefore the students concluded that having the same angle measurements does not ensure congruency. Ms. V explains that the students found one case where (AAA) does not work. Ms. V continues with the following explanation.

Ms. V: Now class this is a special case we call a counterexample. This example is important because it shows that our proposed condition of AAA does not hold.

Casey raises her hand and asks a question,

Casey: Ms. why is a counterexample so important?

(Ms. V smiles and motions to Tia, who has raised her hand it seems in anticipation to give a response to Casey's question).

Tia: Because in mathematics if you have a counterexample it can prove that a statement you made is not true.

Ms. V: Yes, as Tia said, in mathematics if you find one case where a theorem or claim does not hold then you use this case as a counterexample that will prove the claim is false. Casey, do you understand the reason why it is important?

Casey smiles and excitedly, looking up while writing in her notebook.

Casey: Yes I got it; I now see why AAA does not hold because of the counterexample.

Ms. V: Does everyone understand why the AAA condition is not valid?

Students: Yes Ms. (*In unison, nodding their heads in agreement while writing in their notebooks.)*

Joe's Solution

Ms. V reviews the three necessary conditions for congruency. She places a slide summarizing the solution to Slow Joe's problem. Figure 3.14 shows the slide with the three solutions to the problem.

QUESTION: *What is the minimum number of necessary measurements Joe must use to make congruent triangular trusses?*

JOE'S SOLUTION

Use a minimum of three necessary measurements, which include a combination of sides and angles in a specific order to ensure his trusses are congruent
SSS, SAS, or ASA

Figure 3.14: The necessary conditions for solving Joe's problem.

A Fourth Condition

Ms. V suggests to students that their exists a fourth condition. She asked the class "Did any group get a condition that we have not discussed?" None of the groups volunteer. Ms. V then proceeds to write on the board "Ms. V's Conjecture: Angle-Angle-Side". Ms. V explains that two triangles are congruent if we have two angles of one triangle equal to two angles of the other triangle and one pair of corresponding sides are the same. Ms. V emphasizes the following, "Please note the side does not have to be enclosed by the angles hence we write AAS the order is not the same as ASA where the side is enclosed by the angles." Students listen and write in their notebooks.

Proving Congruency

Ms. V explains to the students that the conditions they found are called postulates and they can be used to prove that two triangles are congruent. Ms. V shows another activity on the screen about proving congruency of triangles (see Figure 3.15). The activity presents to students pairs of triangles with equivalent markings for the angles and sides. The instructions state, "Based on the markings provided for each pair of triangles, state which congruency postulate can be used to prove that the triangles are congruent." Ms. V reads the instructions twice to ensure that all students understand what is required to complete the exercise. Ms. V announces that the class will do it together and she will provide a similar activity in the worksheet that will be assigned as homework. In her post-observation interview, Ms. V explained, "I really enjoyed how the last activity for proving congruency went, I really felt that the students understood how to use the postulates when writing proofs. I am looking forward to our next lesson where they will practice writing an entire proof."

The first exercise has two triangles with two corresponding sides and the enclosed angle (see Figure 3.15 a). Ms. V asks the students, "Which postulate proves that these two triangles are congruent? Remember your answer must be supported with reasons, so who has an answer?" Melissa raises her hand and then answers SAS. She points at the diagrams on the smart board and identify the two sides and the enclosed angle that support her use of the SAS postulate. In Figure 3.15 (a), the single and double markings indicate the corresponding sides and angles in the pair of triangles.

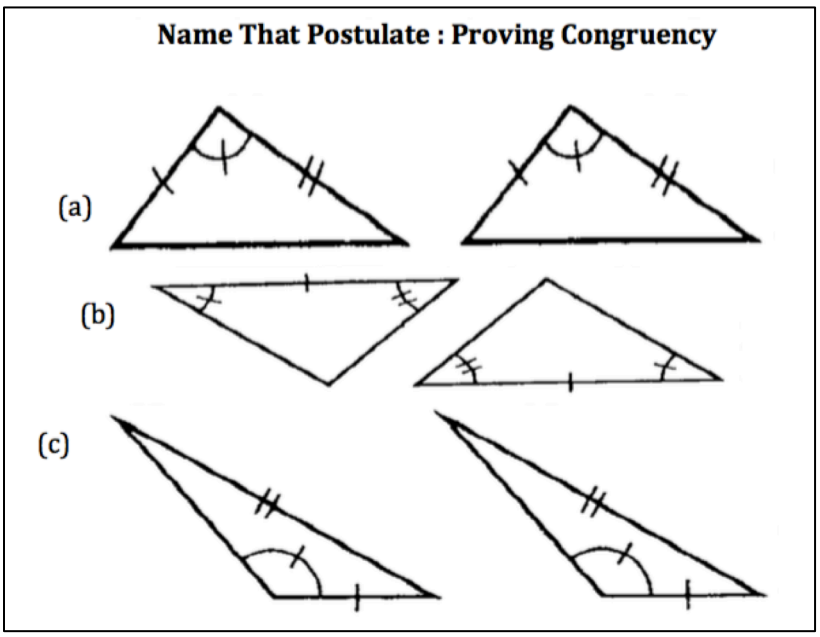


Figure 3.15: Student activity for proving congruency.

Ms. V evaluates Melissa's answer by saying "Well done." She then asks, "Does everyone follow Melissa's explanation and answer?" The students respond, "Yes" and a few nod in agreement. Ms. V projects two more examples and points to each one asking students to say which postulate can be used. Sara answers the second example shown in Figure 3.15 (b). Sara claims that ASA should be used, "because of the corresponding two angles and the side between them." Ms. V smiles and says "Well done" she points to the third example shown in Figure 3.15 (c). Tia raises her hand and then proceeds to give her answer. However, she pauses and then says

Tia: Ms. hold on, this one is tricky

Ms. V: Okay just think about it.

Tia pauses again and then smiles.

Tia: Okay I have it, this one cannot be proven with any of the postulates because it looks like SSA but that is not really a condition.

Ms. V: Okay class Tia is saying that none of our conditions satisfy the indicated corresponding components in this diagram. She is also saying it seems like SSA but this was not one of our conditions. Do we agree or disagree?

Students: *(In unison)* Agree.

Ms. V: Good Job.

The bell rings, it is now 1 hour and 30 minutes later. Ms. V announces, "Please review today's work for our next class." The students get up from their seats, packing up their notebooks. Ms. V walks over to the smart board talking to some students as they walk pass her on their way out to the door.

Profile of Mr. P

Mr. P's school principal recommended him based on his leadership role, mentoring of his colleagues, and years of experience teaching mathematics. During my observations at his school, several junior teachers observed his classes. Through these observations, novice teachers were expected to understand the nuances of everyday activities surrounding mathematics teaching. Mr. P was well-respected by his colleagues. During my data collection at the school, Mr. P spent his free class periods assisting his fellow colleagues with lesson planning sessions. I observed one of these sessions, which included teachers sharing ideas for teaching various content and collaborative planning of lessons for upcoming topics during the semester. In these sessions, Mr. P seemed to the expert in answering questions, providing feedback, and guiding the discussions.

With regard to the teaching of mathematics, Mr. P stated that his main goal was to get his students to talk about their ideas. When I asked Mr. P what strategies do you use to engage your students in reasoning and proof? Mr. P responded:

> Mr. P: Well in my classes I always want the students to talk about the ideas, I try not to tell them the rules and theorems but let them lead the discussions to discover the material for themselves. I achieve this by always starting with some video clip or an activity that could get them talking and interested in what we will be doing on

> that day. I usually collaborate with the other teachers so that we can come up with an interesting entry point for the lesson.

In the above quote, Mr. P achieves his lesson's objectives by generating discussions that allow students to discover the required content for the lesson. Mr. P stated that his lessons would initially begin with an activity or video clip that fostered the type of discussions that lead to the discovery of mathematical ideas. When Mr. P stated, "I usually collaborate with the other teachers so that we can come up with an interesting entry point for the lesson," he seemed to refer to his lesson planning sessions. Mr. P asserted that these sessions allowed collaboration among his colleagues and provided him with useful ideas for starting his lesson. Overall it seemed that Mr. P's leadership and professional experience provided opportunities for mentoring, collaboration, and sharing of instructional ideas among colleagues.

Vignette of Mr. P: Finding the Twin: Defining Triangle Congruency

This vignette describes the first of three lessons where students are introduced to the idea of congruent shapes. Mr. P facilitates group activities along with whole class discussions for students to come up with the congruency postulates. Mr. P leads the students through pattern identification, conjecturing, and writing proofs.

Opening Vignette

The bell rings as students enter the door, walking quickly to their seats. Mr. P walks to the back of the room and pulls down the screen for the projector. He quickly moves to the front while making the following announcement, "Okay students settle down, we are going to look a video clip to begin today's lesson. Remember to sit in your usual groups and you must pay attention to the clip. I will be asking you some questions." Students listen as they take out their notebooks and textbooks from their knapsacks. Some students rearrange their chairs so that they are facing the back of the room where the screen is located. Mr. P closes the door located to the

right of the room and begins the video clips. Students look at the screen as the video begins. Some students make a sound "Shssh" indicating for everyone to keep quiet as the video starts.

Introduction to Congruency Lesson

Mr. P shows a video clip taken from Sesame Street entitled: "Triangles in The Museum" PBS (1984). The video may be found at: https://youtu.be/0ZcKEZO_JqA. In this video an animated character in the shape of a blue triangle is walking through an art museum looking for his twin. While walking through the hallways he says, "Yoo-hoo! Where are you triangle? Are you in the museum? I will find you triangle." In the first room he enters, there is a portrait to the left with a few shapes embedded in the picture. He looks carefully at the various shapes in the portrait. Several triangles blink or flash on the screen but they are either smaller or larger than he is. The blue triangle continues walking throughout the museum saying, "Yoo-hoo! Where are you triangle?" Eventually he finds the shape congruent to him, which is his twin. They greet each other and walk out of the building.

The students watch the video clip. When the clip ends, Mr. P shows it again, reminding students to pay close attention to what is occurring. Mr. P asks the students to describe what occurred in the video. A few students volunteer and respond. One student says the triangle is looking for the same shape. Mr. P responds and asks the student "What do you mean by same shape?" The student responds, "Well I mean triangle that looks exactly like him, with the same shape and size." Another student says that there are other triangles that seem to look the same but were not because they were neither the same shape nor size. Mr. P nods it seems in approval of the two students responses. He repeats the students' explanations and asks the students to reflect on what was discussed.

Classifying Congruent Shapes

Mr. P introduces an activity where he has several different shapes taped to the white board at the front of the classroom. Three are nine shapes, consisting of three triangles, two squares, two rectangles, one circle and one hexagon (See Figure 3.16). Mr. P asks the students to suggest how to classify the shapes.

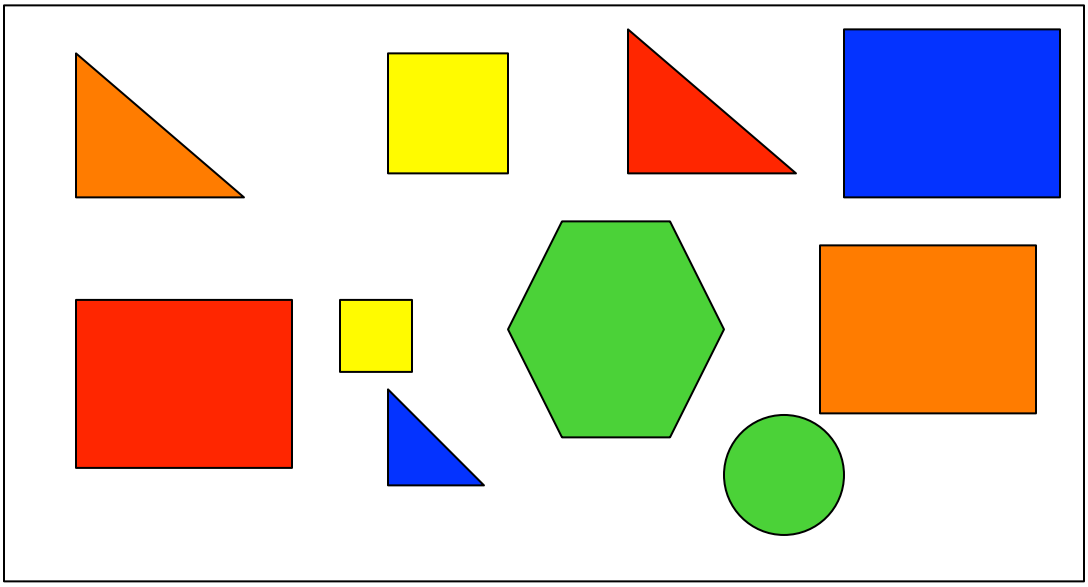


Figure 3.16: Mr. P's classification of shapes activity.

Mr. P divides the white board into columns to represent the different classifications students will use. Each column represents one method and Mr. P asks students to stick the shapes in the appropriate columns. Shane volunteers and classifies the shapes according to size and sticks the shapes with the same sizes into the column. Mr. P labels this column "Size." He tells Shane, "Good job." Mr. P solicits another method. He asks students "What other method could you use to classify these shapes?" Another student, Ava volunteers, and classifies the shapes according to color. Ava places the shapes in the second column, which Mr. P has now labeled "Color." Ava also use some of the shapes from column 1 in her classification. Mr. P addresses

the class and says that it is okay to use from a previous column. Students listen while writing in their notebooks.

Mr. P solicits another method. He asks students "What other method could you use to classify these shapes?" A third student, Jonathan, volunteers and classifies the shapes according to shape. Jonathan places the shapes of the same shape in the third column on the white board. The column has a mixture of triangles and quadrilaterals. Mr. P asks again "How can you classify these shapes according to any common characteristics you notice?" Sean, raises his hand and walks to the white board after Mr. P gestures to him to come forward. Sean further classifies Jonathan's shapes into triangles and quadrilaterals. Mr. P says to Sean "Very good observation Sean" Mr. P now addresses the class "Okay students let us look at this, Sean has modified Jonathan's classification into different types of shapes. Mr. P asks Sean "What made you do this?" Sean pauses for a few seconds and then responds. "Well Sir when Jonathan placed the shapes that are the same, I realized he had triangles, squares, and rectangles. I noticed that the shapes in the column could go further into two groups triangles and quadrilaterals." Dana raises her hands and Mr. P allows her to speak. Dana says, "Sir, the quadrilaterals could go even further into squares and rectangles." Mr. P smiles and then says to the students, "Good job." Mr. P seems pleased with Sean and Ava's active role in enhancing Jonathan's method.

Mr. P explained in his post-observation interview, that this was a good moment in the lesson. When I probed further by asking, "Why was this moment important to you? Mr. P explained, "When the term started and we began having more whole class and group discussions, we agreed that it is okay for another classmate to build on or enhance another student's ideas. I really want the students to feel free to give feedback could help improve an idea or initiate another useful idea."

Defining the Concept of Congruency

Mr. P asks the students, "Does anyone else have another classification we could use?" Delina volunteers to present her solution. Delina classifies the shapes according to those that look the same. Delina explains that in her classification she is looking at the same shape and size. Delina places the shapes that are identical in the fourth column. There are three triangles, two squares, and two rectangles.

Mr. P asks the students "What do you notice about the shapes in this final classification?" Ava responds "Sir they have the same shape and size." Nick raises his hand and says, "They also have the same area, perimeter, sides, angles, and lines of symmetry." Mr. P nods it seemingly in approval for Ava and Nick's answers. Mr. P asks, "What can we conclude here considering the words you just used in your descriptions." Before anyone could answer, he says, "In Mathematics, we say that when objects have the same shape and size they are congruent. What is the word class?" Students repeat in unison "Congruent!" Mr. P moves the shapes from the various columns on the white board. He erases the columns and writes a definition for congruency. Students listen and take notes.

A Physical Representation of Congruency

After writing the definition for congruency, Mr. P asks for two volunteers to stand. Two students volunteer, they stand at the front of the room facing the other students. Mr. P asks the students to identify objects on the two volunteers that may be congruent. Students suggest the monogram on their school uniform. They also suggest that one of the student's pair of earrings, which are circular, are congruent. Mr. P acknowledges these suggestions; he also proposes that they consider any congruent objects in the classroom. Students suggest the windows, the textbooks; the two rectangular fixtures in the ceiling are also congruent. Mr. P says, "Very good

observations." He then asks the students to read aloud the definition on the white board. The students read in unison "Two objects are congruent if they have the same size and shape."

Connections to Related Topics

After the students read the definition. Mr. P makes a connection between congruency and a previous lesson on Geometry transformations. Mr. P explained during the post-observation interview that students learned about reflections, enlargements, and rotations in a previous lesson. He stated, "I think it was important to make the connection between what they previous learn and the concept of congruency." Mr. P's comment seemed to suggest the reasons for referencing this topic during the discussions on congruency.

Mr. P shows two squares a smaller and a larger one. He explains that the larger square can be produced from the enlargement of the smaller square. He asks the students, "in this case of enlargement, are they congruent? Why or Why not?" Several students shout "No!" Mr. P points to Antonio and asks him to explain why.

> Antonio: Sir, they are not congruent because the image is not the same size although it is the same shape as the object.
>
> Mr. P: Okay, so what Antonio is saying is that the two objects are not congruent because the image is not the same size. So Antonio, what does this tell us about enlargement?
>
> Antonio: Well, congruent objects must be the same size and shape and with enlargement the object gets bigger.
>
> Mr. P: Good job! Now let us look at this diagram here.

Mr. P places a picture on the white board shown in Figure 3.17. Mr. P points to the mirror line and explains that the blue object results from a reflection of the red object.

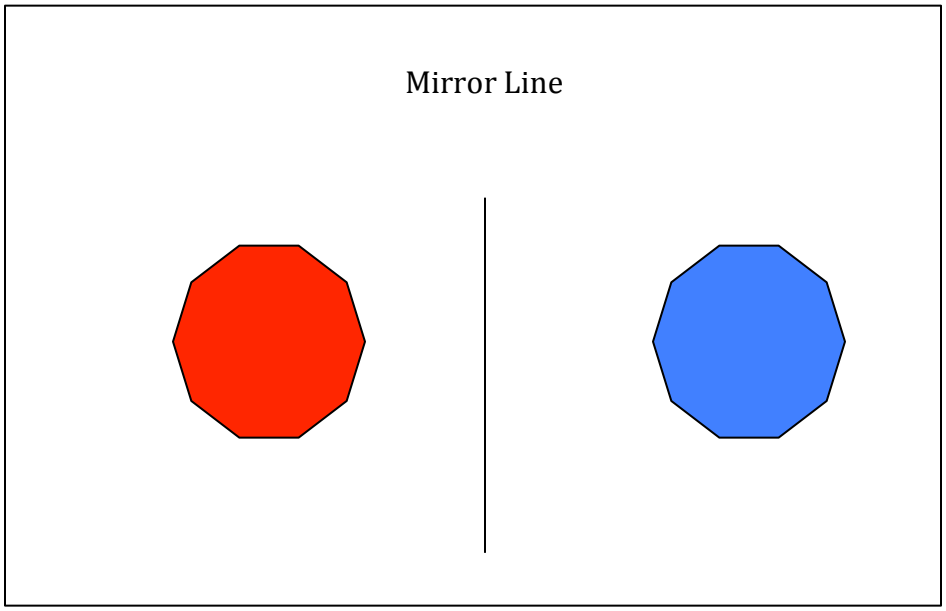


Figure 3.17: Example of reflection of geometrical objects.

Mr. P points to the objects and asks, the question, "Do you think these two objects are congruent? Explain?" Students unanimously say, "Yes," Sophie volunteers and Mr. P allows her to explain.

Sophie: Well, Sir in a reflection the size and shape are kept the same so they will be congruent, unlike with enlargement where the image was not the same size as the object.

Mr. P: Hmm so you are saying that reflection produces congruent objects while enlargement do not.

Sophie: Yes Sir according to our definition of congruency, reflection will give congruent objects.

Mr. P: Good job.

Mr. P moves to his desk and begins distributing packets to the students

Identifying Congruent Triangles: Group Activity

Mr. P walks around the room distributing packets to several groups of students. The packet contains five triangles. Students are expected to use their measuring instruments such as a protractor and ruler to measure the triangles and then select the congruent triangles. Figure 3.18

shows the five shapes given to the students. Each triangle is numbered and each groups gets the same shapes in their respective packets. After selecting the congruent triangles, students are expected to determine how the shapes are congruent. Mr. P explains the instructions of the activity and then asks students to look at the packet on their desk with several cut outs of triangles. He then asks students "What do you notice about the congruent triangles? How can you determine if two triangles are congruent?"

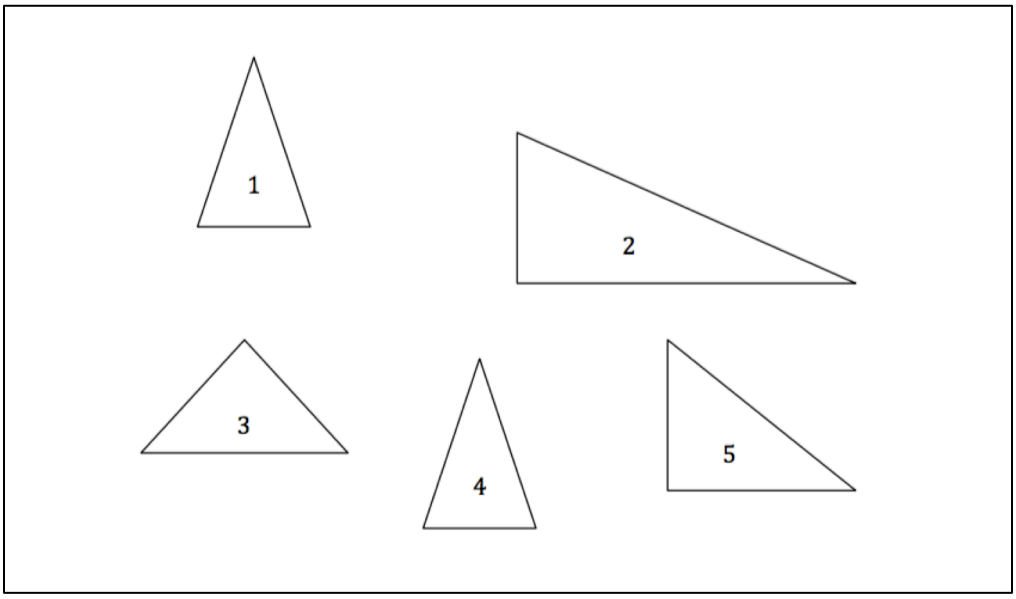


Figure 3.18. The five triangles students use for the activity: Identifying Congruent Triangles.

Students begin to work in groups on the activity. There are eight groups of four to five students. The students work on their assignment for about five minutes. Mr. P walks around the classroom observing the discussions among the groups during the initial eight minutes. After visiting three groups, he makes a general announcement. "Students remember in your group you must come up with reasons for your solutions, we do not want answers only, please remember to back up your statements."

Students listen and nod in agreement and then continue working once Mr. P completes his announcement. About six minutes later, Mr. P asks the groups to listen now and respect each

group's contribution. He says to the students, "Sit up no talking and direct your attention to whoever is reporting their results." He then asks for a group to volunteer and report their solution. Devin raises his hands quickly indicating that his group wishes to present their solution. Mr. P says, "Silence we have the group of boys to the back ready to present. Please listen." When the class becomes completely silent, Devin begins to speak.

Devin: Well Sir we agreed that triangles 1 and 4 are congruent.

Mr. P: Okay, we have these guys claiming triangles 1 and 4 are congruent. Okay everyone follows?

Students: Yes Sir.

Mr. P: Okay which measurements did they use to determine that the triangles are congruent?

Devin: Well, the angles

Mr. P: What about the angles? Remember you need to back up your answer so that everyone can understand your answer.

Devin pauses looks around, another boy in the group whispers to him. Devin nods, smiles, and then speaks,

Devin: Okay, what we did we measured the angles and they are all the same so we decided that if the three interior angles are the same in a pair of triangles, then they are congruent.

Mr. P: Okay, this group proposes that when all three angles are the same then the triangles are congruent. Okay, everyone follows this? Do you all agree?

Shane raises his hands; Mr. P acknowledges him and gestures to him to ask his question.

Shane: Sir for some reason I feel it may not work all the time. Is it possible that two triangles could have the same angles but different sizes?

Mr. P: All right we have a question here. Firstly, Shane feels that this group's claim will not hold all the time. Shane is asking if it is possible that two triangles could have the same angle measurements but is different in size?

Students begin to whisper among themselves, Sharon raises her hand and begins to speak.

Sharon: Well Sir I think that it can happen like in enlargements.

Mr. P: Okay good we are on to something, let us look at this for a moment

Investigating the Case of Angle-Angle-Angle

Mr. P takes two right triangles of different sizes (see Figure 3.19) and asks the students which angles are the same. The students say the right angles and the two other angles look the same. Mr. P prompts them further by saying "Well, in fact, when I measured them earlier the angles are the same. However, are they congruent?" The students say, "No, they are not the same size." Sharon says," Sir it looks like an enlargement and it looks like the angles are the same but they are not the same size" Mr. P smiles and then says, "Yes."

Students begin to chuckle, the group members in Sharon's group gives her a high five.

Mr. P: Devin's group suggested AAA but we showed that this does not ensure that any pair of triangles is congruent. Do you follow what I am saying?

Students: *(In unison)* Yes Sir.

Mr. P: Okay, what do you need to modify in their claim?

Raquel raises her hand, Mr. P points to her and she begins to explain.

Raquel: Well Sir we cannot use that we could use the sides.

Mr. P: Okay did everyone hear this? Raquel is suggesting that we could use sides. What about the sides should we consider?

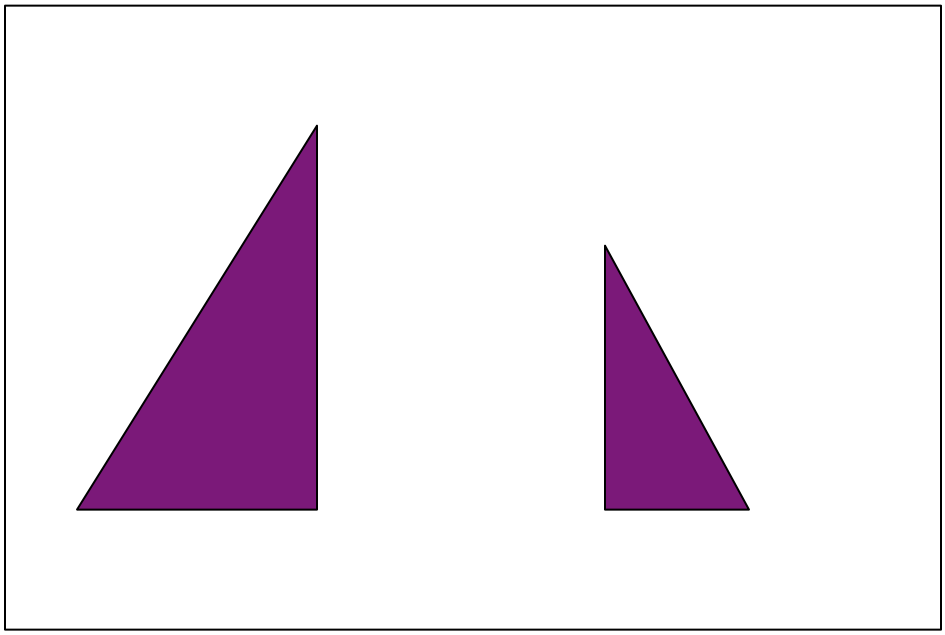


Figure 3.19: Example of non-congruent right-angled triangles.

Jessie waves his hands to get Mr. P's attention.

Mr. P: Okay Jessie, let us hear what you want to say.

Jessie: Okay Sir as Raquel said the angles will not work but all the sides must be the same to ensure the same size and shape.

Mr. P asks Jessie to come to the white board and label the sides of two triangles on the white board that are congruent. Jessie walks to the white board and labels the sides of the two congruent triangles with corresponding markings on the sides (see Figure 3.20). Mr. P notices that two of the sides in Triangle 1 have the same single stroke. He asks the following,

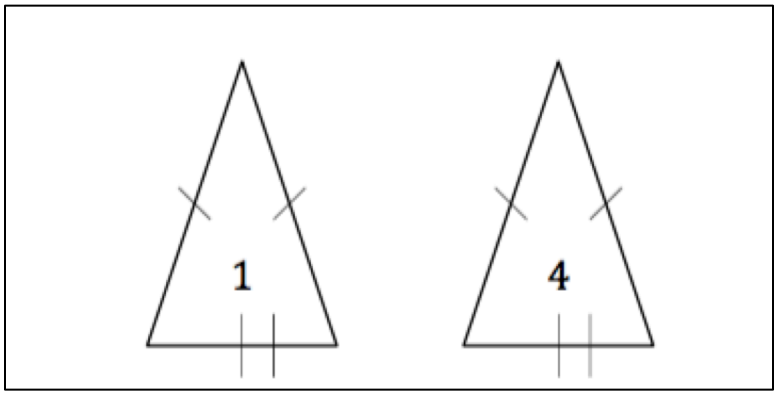


Figure 3.20: Students' solution for identifying congruent triangles activity.

Mr. P: Okay I notice that you have the same markings on two of the sides in triangle 1 and 4. Why did you do that?

Jessie: Well Sir from our measurements we found that the two sides are the same about 7.5 cm. Therefore triangles 1 and 4 have two corresponding sides that are the same length, thus they are isosceles triangles, and they are both congruent.

Mr. P: So you are saying that triangles 1 and 4 are isosceles triangles.

Jessie: Yes and the corresponding sides are equal so this makes them congruent.

Mr. P: So what about the third side?

Jessie: Well we measured this side *(Jesse points to the third side in triangle 1)* and it is the same length as the third side here *(Jesse points to the third side in triangle 4)*, which is 5 cm. This is why I put the same double markings on the corresponding sides.

Mr. P: Wait a minute say that last word again!

Jessie: Corresponding?

Mr. P: Yes, class listen the sides are corresponding. Do you get that word?

Students: Yes Sir.

Mr. P: What is the word again?

Students: Cor-res-pon-ding! *(The students emphasize the syllables as Mr. P writes the word on the white board and points to the word.*

Mr. P: Okay good job let us write a definition and I want you to tell me what to write because it is your definition.

Mr. P explained in the pre- and post-observation interview that one of his strategies for teaching was giving students the autonomy to come up with their own definitions. After the class has reached an agreement on a final definition or formula, he would usually allow them to narrate the definition while he writes. I asked him, Why this is important? Mr. P explained, "to give them ownership of what they learn." Mr. P clarified further "this helps students see the value of what they are learning rather than me telling them directly." Mr. P claims that some

students on seeing the formula in their book after the class would tell him their definition or formula is the same but their version is easier to remember because they wrote it.

Mr. P writes a definition based on the discussion. He allows the students to dictate the definition, "Two triangles are congruent if three sides of one triangle are equal to the corresponding three sides of the other triangle." After writing the conjecture, Mr. P reads it aloud twice. Students listen while writing in their notebooks. Figure 3.21 shows students definition of the side-side-side postulate.

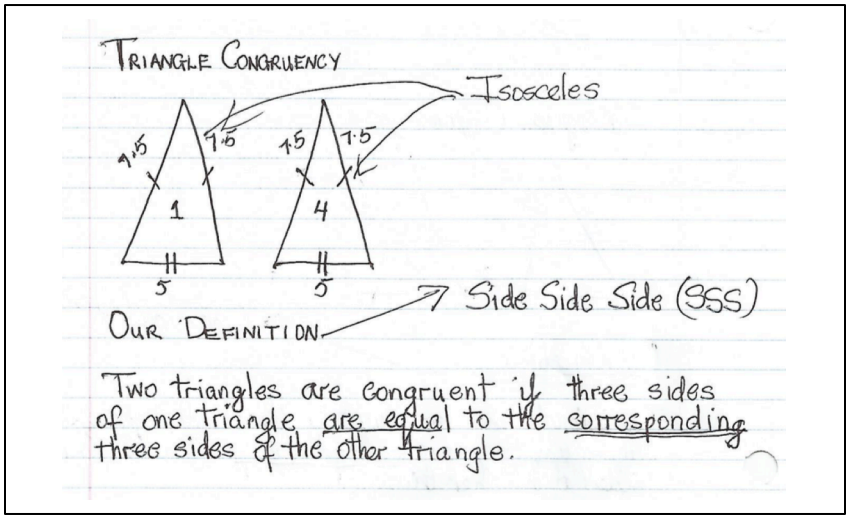


Figure 3.21: Sample of students' definition of side-side-side congruency postulate.

Mr. P gives each group a worksheet with practice questions. The worksheet comprises of questions from the textbook and CSEC past examination questions. Students are expected to practice identifying congruent triangles. He assigns questions 7 and 8. Figure 3.22 shows question 7. As shown in Figure 3.22, in answering question 7, students are expected to determine whether triangles *ABC* and *ADC* are congruent. Students must also provide supporting reasons explaining their claims about the triangles congruency. Students begin working on the worksheet in their groups. Mr. P reminds the students that they must explain why the triangles are congruent.

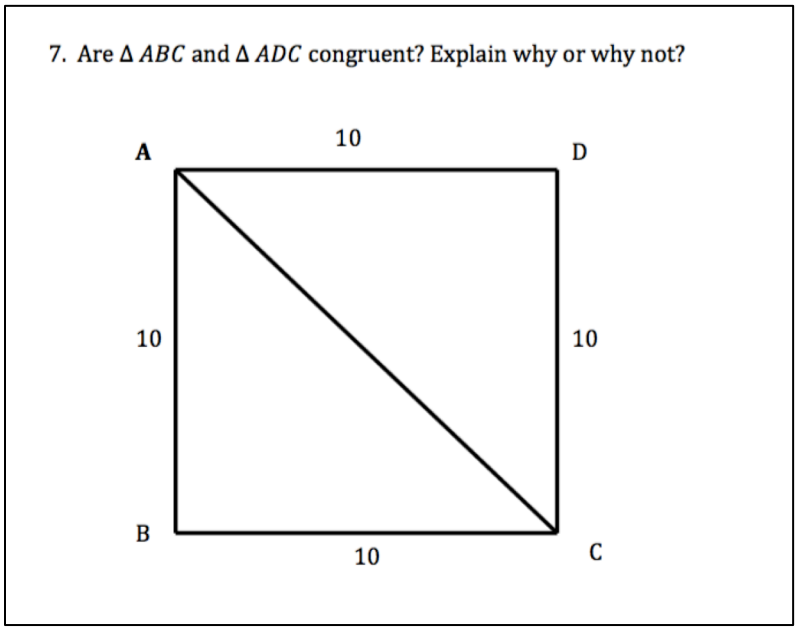


Figure 3.22: Question 7 from Mr. P's congruency worksheet.

Mr. P monitors the groups by only observing their discussions. He also selects one of the groups, which did not have an earlier opportunity to share their answers during the lesson, to report their solutions. After, about 8 minutes, Mr. P calls the class to order by saying, "Okay everyone stop writing and talking, sit up, and let us pay attention to the group over here." Mr. P points at a group to the left of the room. He motions to them to begin their presentation. Randy stands and begins to report but the class is noisy. Mr. P stops Randy and reminds the class of the rule about giving their undivided attention when a group is reporting.

Mr. P: Now everyone remember when someone from a group is reporting you must not talk but listen to their explanations. If you have a question raise your hand and direct it to the person but you must all be quiet and listening.

When the students become completely quiet Mr. P asks Randy to proceed with his explanation.

Randy: The two triangles *ABC* and *ADC* are congruent because…Well from the diagram we see that *AB* = *AD* because their measurement is 10. *BC* = *DC* because they are both 10.

Mr. P: So what does this mean?

Randy: Well this means that two of the sides are corresponding.

Mr. P: Good! So Randy has said that the triangles are congruent because the side $\overline{AB}$ corresponds with $\overline{AD}$ and $\overline{BC}$ corresponds with $\overline{DC}$. Okay, what else will make them congruent?

Randy looks at the members of his group, he nods at Penny. Penny smiles, stands, and begins to speak as Randy sits.

Penny: The diagonal $\overline{CA}$ is the same for both triangles because it is like the line of symmetry in the quadrilateral.

Mr. P: The diagonal $\overline{CA}$? What do you mean?

Penny: Yes what I mean is that the side $\overline{CA}$ is the same in triangles *ABC* and *ADC*, so because it is the same, all the sides of triangle *ABC* correspond with those of triangle *ADC*. They are equal! Therefore we can conclude they are congruent!

Mr. P: Okay so this group is saying that because all the sides of the triangle *ABC* correspond with and are equal to the sides in *ADC*, the triangles are congruent. Does everyone follow this?

Students: *(In unison)* Yes Sir.

Mr. P: Okay let me write the proof of this on the board.

Mr. P writes the proof of the congruency of the triangles in question 7 on the white board. He explains the various symbols in the proof. Figure 3.23 shows a sample of a student's notes containing the proof written on the white board.

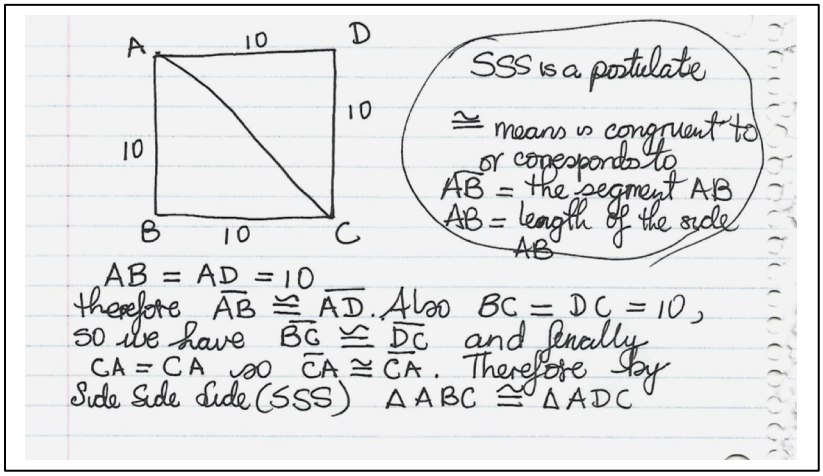


Figure 3.23: A student's notes of Mr. P's proof of question 7 on the congruency worksheet.

While Mr. P writes the proof on the white board he explains that Side-Side-Side is an important result you must use to prove that two triangles are congruent. Mr. P says "SSS is what we call a postulate of congruency, what I mean is that it is an accepted fact that you can use to prove congruency of triangles. Okay." Mr. P continues writing the proof and he explains and reviews some notations. He explains, "This symbol ≅ here (pointing to the symbol he has written in the proof) means is congruent to, okay." He also reminds students of the notations they used in another topic of Geometry. In particular, he reminds them of the different notation for a line segment and for the length of a segment. Mr. P asks, "Does everyone follow this? "Students reply in unison "Yes Sir." Figure 3.23 shows that the student circled the additional comments Mr. P provided about the mathematical notations and meanings of symbols in the proof. The student seemed to highlight these comments as a possible reminder of the mathematical practice of using certain notations when writing future proofs for congruency.

Mr. P begins to walk around the room observing the students writing in their notebooks. He moves on to question 8 (see Figure 3.24 below) and asks "Which of these triangles are congruent and why?" As shown in Figure 3.24, the geometrical object contains three triangles *ABC*, *ADC*, and *ADE*. Students are required to state which triangles are congruent. Students must provide supporting reasons explaining their claims about congruency.

Mr. P begins to walk around the room observing the students writing in their notebooks. He moves on to question 8 (see Figure 3.24) and asks "Which of these triangles are congruent and why?" As shown in Figure 3.24, the geometrical object contains three triangles *ABC*, *ADC*, and *ADE*. Students are required to state which triangles are congruent. Students must provide supporting reasons explaining their claims about congruency.

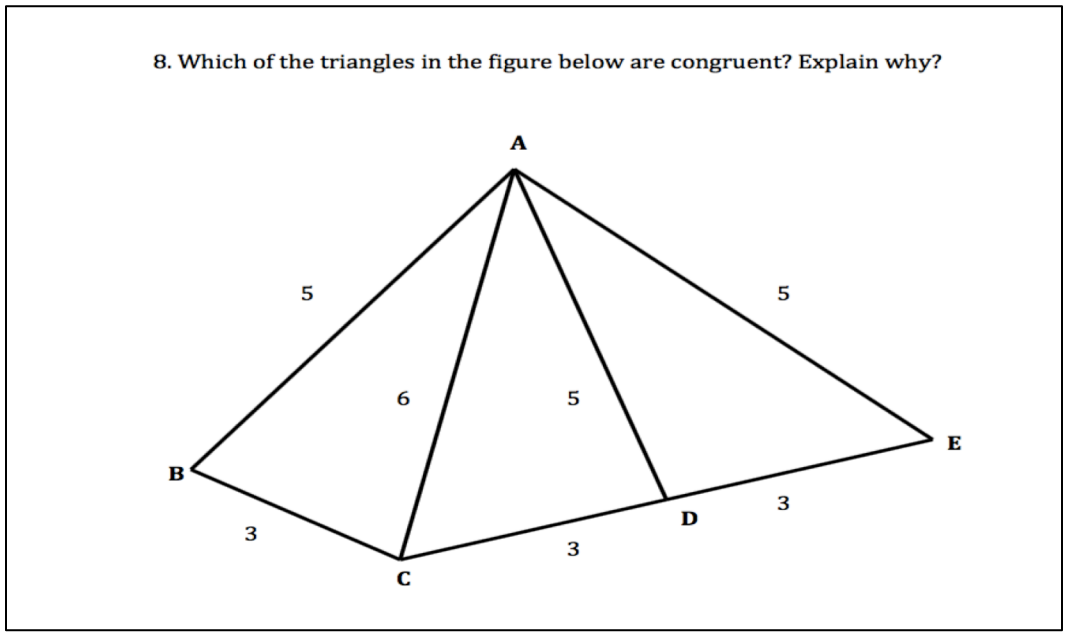


Figure 3.24: Question 8 from Mr. P's congruency worksheet.

Mr. P appoints another group, which have not reported yet to give their answer. Delina stands

Delina: Well we found that triangles *ABC* and *ADC* are congruent because *AC* is the same in both triangles and *AB = AD = 5* and *BC = CD = 3*. Therefore by SSS we can say that triangles *ABC* and *ACD* are congruent.

Mr. P: Okay good, Delina is saying that from her group, triangles *ABC* and ADC are congruent because of three corresponding equal sides. Okay what about *ADE*?

Delina: Well the first two triangles are congruent but the second and third are not because all their sides are not the same.

Mr. P: Okay which ones?

Delina: Well *AB = AD* and *BC = DE* but *AC = 6* and is not equal to *AD* which is *5*.

Mr. P: Yes good job. Does everyone agree with this explanation and answer?

Students: *(In unison)* Yes Sir.

Mr. P: Okay let us move on a bit because the bell will ring soon.

Mr. P then asks the students to return to the earlier group activity on their desk. Mr. P announces,

"Okay now using your measurements for triangles 1 and 4. How else besides three sides can you

determine that two triangles are congruent?" Students continue to work on their activity. Mr. P walks around the room observing the various groups' discussions. After five minutes, Mr. P appoints another group to report. In the post-observation interview in response to my question "What went well in the lesson today?" Mr. P explained that he purposefully selected the group to report their findings because he believed their idea could generate the type of discussion he wanted to achieve. Figure 3.25 shows the student's measurements for the triangles 1 and 4.

Mr. P asks the students to sit up, stop writing, and listen to the group's presentation. A volunteer from the group stands and begins speaking.

Ava: Well in our group we realized that if you have two isosceles triangles with same sides *7.5* from their measurements and with the same angle of 35^0 between the sides they would be congruent *(see Figure 3.25)*.

Mr. P: Hmm so you are saying that if we use the two corresponding sides with the same angle between them we will get congruent triangles.

Ava: Yes that's what we did we used two sides which were both 7.5 cm and the angle of 35^0 between them.

Mr. P: Okay good job. Does everyone follow that groups' explanation?

Students: Yes Sir? (*Some of the students answer while others nod their heads it seemed in agreement while writing in their notebooks*)

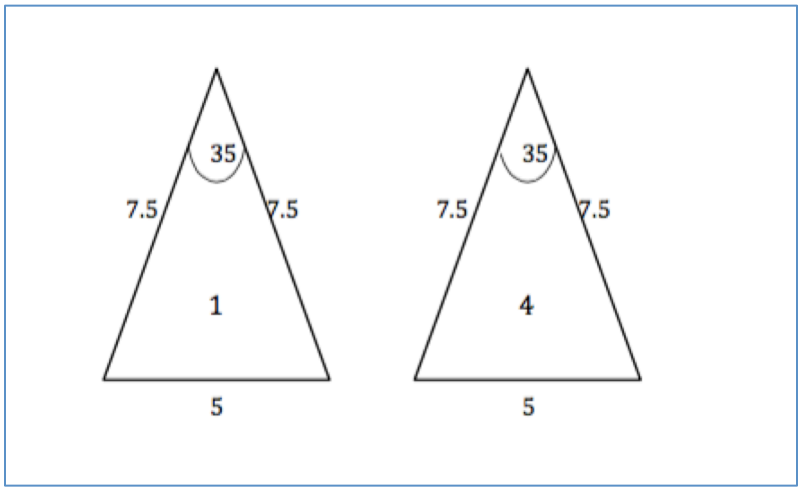


Figure 3.25: Students' measurements for Side-Angle-Side congruency postulate activity.

Mr. P: Okay in general how could we write this new postulate for congruency?

Shane raises his hand. Mr. P acknowledges his hand.

Mr. P: Do you have a question?

Shane: Yes Sir.

Mr. P: Okay, go ahead. Now students listen, you need to listen to his question.

Shane: Okay Sir in our group we got the same thing but we realized that if you do not use the angle in between the sides it would not work.

Mr. P: Oh, so you are saying that the angle must be enclosed between the two sides.

Shane: Yes.

Mr. P: So you are saying that any two sides and any angle would not work?

Shane: Yes, the angle must be the one between the two sides.

Mr. P: Okay class do you get this?

Students: *(In unison)* Yes Sir.

Mr. P: Okay so let us write up this Side-Angle-Side (SAS) postulate

Mr. P draws the students' previous particular example of triangle 1 as shown in Figure 3.25 on the white board to prompt the students. Mr. P says, "Okay students using the diagram here with corresponding sides and angles labeled. How can we write this group's conjecture Side-Angle-Side?" Students begin to dictate the conjecture as Mr. P writes on the white board, "Two triangles are congruent if two sides and the included angle of one triangle is equal to the corresponding two sides and the included angle of the other triangle." Students listen as Mr. P reads the SAS postulate aloud twice. The students continue writing in their notebooks.

Assessing Students' Understanding of Side-Angle-Side Postulate

Mr. P moves to an example of two triangles *ABC* and *PQR*. Figure 3.26 shows the question Mr. P used to assess students' understanding of the SAS postulate.

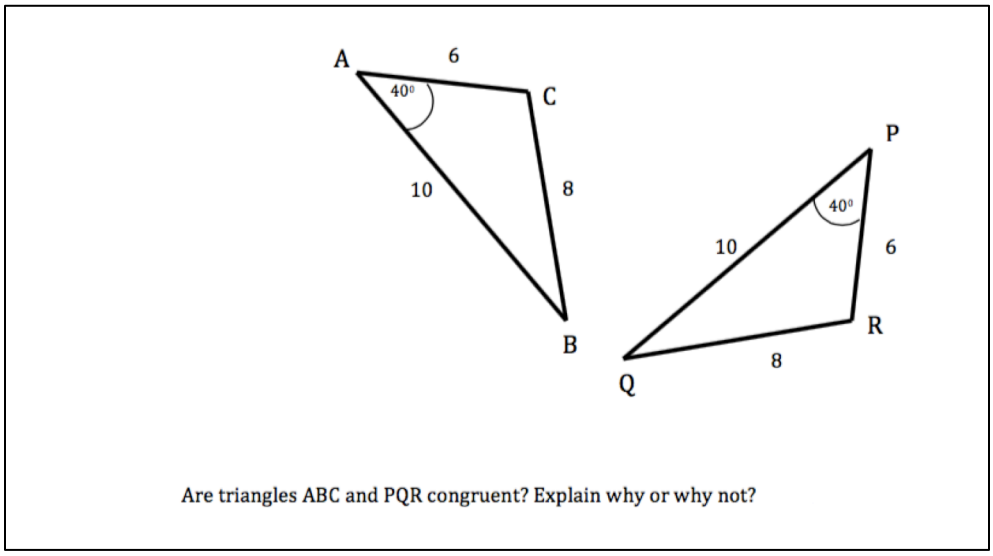


Figure 3.26: Mr. P's exercise for assessing students' understanding of Side-Angle-Side postulate.

The question contains two triangles with measurements for the three sides and one angle. Mr. P draws the diagrams with the respective measurements. Mr. P points to the diagram and asks, "Are these two triangles congruent? How many of you say yes? Put up your hands." About three quarters of the students put up their hands. Mr. P selects one student and says

Mr. P: Okay Trisha what do you think?

Trisha: well yes because the angles *A* and *P* are the same and…

Trisha pauses and does not continue to speak.

Mr. P: Okay good Trisha. Is that all? Anybody else?

Zafiyah raises her hand and Mr. P acknowledges her and says:

Mr. P: Okay Zafiyah help Trisha out here, what else?

Zafiyah: Well based on what we just defined for SAS, the two sides $AB = PQ = 10$ and $AC = PR = 6$ and angles A and P are both 40^0.

Mr. P: So what you are saying is that the two sides and the included angle are the same in both triangles. So what do you conclude?

Trisha and Zafiyah: *(speaking together)* they are congruent!

The bells rings and students get up from their seats quickly and begin to exit the room while some students pack up the textbooks on their desk. Some students are talking with Mr. P as he erases the white board.

Vignette of Ms. M: Applying the Postulates of Congruence

This vignette describes the second of two lessons where students participate in a group work activity to prove congruency of triangles. Ms. M. guides the students in their various groups as they investigate the congruency of triangles. Student use the provided materials of straws, rulers, and construction paper to construct and draw triangles. Students are expected to use the different congruency postulates to determine whether the constructed triangles are congruent. Students also use the postulates to prove congruency. In the previous lesson Ms. M taught the congruency postulates and students participated in some group activities taken from Toolsie (2009). In these group activities, students discovered the theorems about the sum of interior angles of triangles and quadrilaterals. Following this, Ms. M led a teacher directed recitation about the postulates for congruence. Ms. M also reviewed how to measure angles with a protractor in preparation for the second lesson.

Profile of Ms. M

Ms. M was recommended by the head of her mathematics department due to her experience, formative training in mathematics education, and her approach to classroom management. During my observations at her school, Ms. M offered assistance to her colleagues for planning their lessons. Among her colleagues, Ms. M was known for her innovative

approaches to teaching content. Ms. M asserted this when I asked: What teaching strategies do you implement to engage your students in reasoning and proof? Ms. M explained, "I use classroom discussions and a lot of group work to allow the students to share their ideas." Ms. M stated that her pedagogy was based on research-based theories she learned during her education. Other colleagues usually sought her for assistance in their lesson planning and advice on their pedagogical approaches. Ms. M claimed that she used several external online resources to provide ideas for her lessons. However, she said that she always tried to create her own lessons using references to pop culture and her students' interests. Among her colleagues Ms. M was also known for her effective classroom management, especially in relation to establishing expectations for students that were labeled as challenging by other teachers. As a result she would be assigned the classes with the more difficult disciplinary problems. Yet, Ms. M was able to establish positive relations with the students and develop a productive environment.

Opening Vignette

The classroom buzzes with activity as students walk to their seats taking out their notebooks. Ms. M walks to the chalkboard and writes the heading "Congruency Postulates: How to prove congruency." The bell rings and Ms. M addresses the class, "Okay today we are going to use the congruency postulates we discussed previously to prove triangle congruency." Students stop talking among themselves and listen to Ms. M's opening announcements. Ms. M then continues addressing the class, "Now could you tell me what were the congruency postulates we learned?" A few students raise their hands and Ms. M acknowledges Kathy's hands and motions to her to begin sharing her answer.

Ms. M: Okay Kathy, tell me one.

Kathy: Miss SAS.

Ms. M: Okay what does that mean?

Kathy: Miss it means side angle side, and this is when you have a two triangles that have two corresponding equal sides with the angle between them also equal.

Ms. M: Good job! Now as Kathy says there are two corresponding sides with a corresponding angle between them.

Ms. M writes SAS (Side-Angle-Side) on the chalkboard. Students continue writing in their textbooks. Ms. M then points to Ruth.

Ms. M: What is your answer?

Ruth: Well we have Side-Side-Side (SSS), which has all three sides corresponding.

Ms. M nods her head and says "Good job" and then writes the acronym SSS with the words Side-Side-Side in parentheses next to the acronym on the chalkboard.

Angle-Angle-Side Postulate

After writing the previous postulate on the chalkboard, Ms. M walks around the room as students continue writing in their notebooks. She then walks to the front of the class room again and asks the question "Can someone give me another one?" Dale raises his hands and answers, he says "AAS."

Ms. M nods and writes on the chalkboard the acronym AAS and then asks,

Ms. M: What does that mean? Remember class you need to explain!

Dale: Well it means Angle-Angle-Side and this is when you have two corresponding angles are equal and one side also equal.

Ms. M: So Dale is saying that AAS means two angles are equal and one side is also equal.

Ms. M writes the acronym with its meaning on the chalkboard. Students continue to write in their books. Morgan raises her hand Ms. M points to her,

Ms. M: Do you have a question?

Morgan: Yes I have a question about the last one you wrote on the chalkboard.

Some students begin to talk to each other. Ms. M says, "Quiet everyone! Let us listen to Morgan's question, remember we need to allow everyone to either comment or ask a question and you must be listening and paying attention." The class becomes quiet and Morgan begins to speak,

Morgan: Well doesn't it have something like ASA also where the S is in the middle?

Ms. M: So Morgan is asking if there exists an ASA postulate? Does anyone have an answer for this?"

Oliver raises his hand, Ms. M nods at him saying

Ms. M: Okay Oliver, go on and explain.

Oliver: Well yeah this is the case like the one Dale talked about but instead the side is between the two angles.

Ms. M: Okay so what is the difference? Explain

Oliver: Well in AAS the side is not enclosed it will be a side that is not between the two angles while in ASA the side must be between the two angles.

Ms. M: Morgan, do you follow what he is saying?

Morgan: Oh yes I understand it now (*Morgan nods her head).*

Ms. M writes the acronym ASA with the words "Angle Side Angle" next to it. Ms. M then asks. "Does everyone understand the difference between these two? "Students respond, "Yes Miss" and some nod their heads it seemed in agreement while writing in their books. During her post-observation interview, in response to the question "What do you think well in the lesson today?" Ms. M explained, "I really felt good when Morgan asked the question because it helped us see the difference between those postulates and it gave other students another opportunity to explain it to the class. They value when one of their classmates take over (laughs) well I mean teach them. I am only there to guide the discussion during those times."

The Case of Angle-Angle-Angle

Ms. M then goes on to say, "Well remember there is not AAA". Noel raises his hand and ask, "Miss why?" Several students say "Yeah Miss Why?" During the post-observation interview, Ms. M talked about her students having the right to ask her questions. Ms. M explained,

> Ms. M: When we are doing proofs, they know that it is okay to ask me questions to help them understand. I don't want them just to accept everything I say. It is good for them to inquire and get supporting reasons for claims. They expect the same thing I always tell them to do, back up your claims.

Ms. M then walks to the chalkboard and writes AAA and then says, "Okay remember some weeks ago during construction of triangles we drew two triangles with the same angles and they were different sizes?" The room is silent while students begin to turn the pages in their notebooks to find the construction they completed in a previous class. Ms. M reminded the students of a previous group activity they completed a week ago. In this activity, the students discussed the case where they constructed two triangles having the same angles, 20^0, 80^0 and 100^0. Students were required to use their protractor and ruler to measure and construct the two triangles in their group.

After about one minute, some students begin to say, "Yes Miss we found it"
Ms. M then asked "Were the triangles the same " Students say "no they were different." Ms. M then asked, "Are they congruent then?" the students responded unanimously "No Miss" Ms. M then asked, "Well does AAA work?" Students responded "No Miss," and continue writing in their books. Ms. M. then erases the acronym AAA that she previously wrote on the chalkboard before their discussion and says, "Let us move on."

Launching the Group Activity

Ms. M takes up a stack of papers on her desk and distributes them to the class. She makes the following announcement, "Okay we are going to use these postulates in the group activity I

am distributing, and I would like you to go into your groups." Students get up from their seats and rearrange the furniture to sit in their groups of either five or four students. There are 6 groups, 4 groups of five students and 2 groups of four students.

After 3 minutes the groups are settled and Ms. M asks them to begin working on the activity. She makes the following announcement "Remember everyone must contribute to the group, if you don't understand ask your group members and you must come up with one solution to hand in at the end of class." Students listen and then begin working on the activity.

Discovering Congruent Triangles Activity

Figure 3.27 shows the first part of the group activity entitled "Discovering Congruent Triangles Activity." In this activity, students are required to use drinking straws, protractor, ruler, and construction paper or cardstock to investigate the congruency of a pair of triangles. Students will construct the triangles by cutting the straws into the following lengths (a) two straws 8 centimeters in length, (b) two straws 11 centimeters in length, and (c) 2 straws 5 centimeters in length. After cutting the straws students need to use 3 straws of different lengths to form a triangle on the cardstock paper as shown in Figure 3.27.

Discovering Congruent Triangles Activity

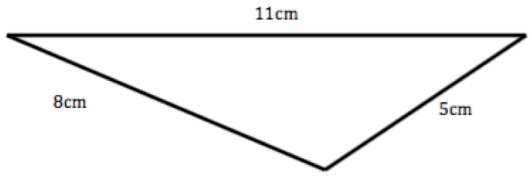
Objectives; Understanding congruent triangle postulates and theorems using inductive reasoning

Materials needed: straws, protractor, ruler, and construction paper or cardstock.

Groups: Divide into groups of 4 to 5 students or pairs.

Have students cut straws into the following lengths:
2 straws 8 centimeters in length
2 straws 11 centimeters in length
2 straws 5 centimeters in length

Part 1
Have students put the 3 straws of different lengths together to form a triangle as shown



2. Form another triangle with the other set of straws.

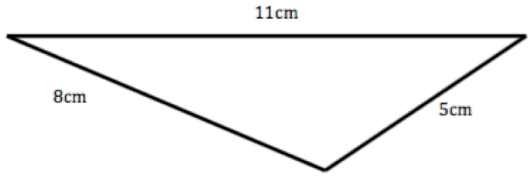


Figure 3.27: Ms. M's class activity on discovering congruent triangles.

On completing the first triangle, students are required to form another triangle with the other set of straws. After completing the second triangle, students are required to use their protractor to measure the interior angles of the two triangles and then discuss the relationship between the angles of each triangle. Students are also required to determine whether the pair of triangles is congruent.

Students work on the activity for about 8 minutes. Ms. M walks about the room visiting each group. After stopping at the second group she reminds students that they can draw the triangles instead of using the straws if they wish. Students continue working and Ms. M

continues visiting each group. In some cases she answers questions when asked by a group. After visiting the fifth group, one of the members, Darcy asks her the following question.

Darcy: Miss do we need to have the triangle in the same orientation as shown in the handout?

Ms. M: Hmm, well what do you think?

Darcy: Well I think that it could be in a different position and we could see if they are congruent.

Ms. M: Okay what does your group think about this?

Wendy, another member of the group responds

Wendy: Well Miss we all came up with this at the same time because we started to draw the second triangle in a different position.

Figure 3.28 shows Wendy's drawing. Wendy shows Ms. M her drawing with triangles *QRS* and *ABC*. Wendy points at the second triangle ABC while Ms. M looks at the drawing.

Miss M: Okay good. So why did you do this?

Wendy: Well we remembered that in class we talked about congruent triangles having the same size and shape and not really same orientation so it might be interesting to draw it in a different position and see if this really holds.

Ms. M: Hmm this seems interesting, try it and see if it will work.

Darcy: Well Miss when they are in the same position we got that they are congruent, so we will try in the different position and see if they are also congruent.

Ms. M: Good, could I share this with the class?

Group members: Sure!

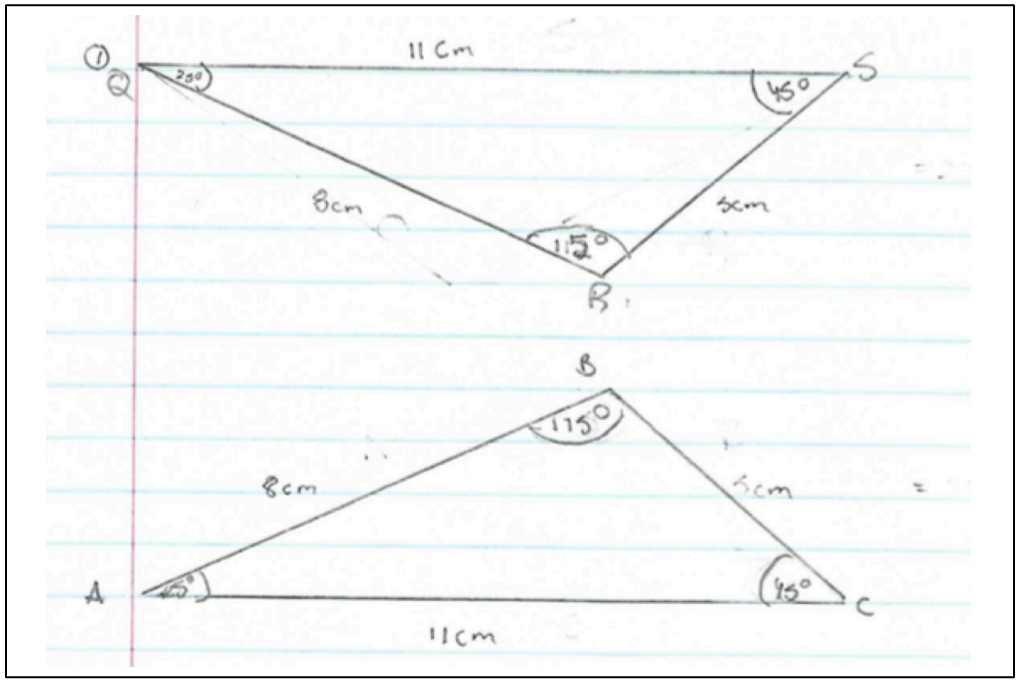


Figure 3.28: Wendy's drawing of congruent triangles in a different orientation.

Ms. M walks to the front and addresses the students,

> Ms. M: Okay everyone we have an interesting idea here from group five. They are going to draw their second triangle in a different orientation than the one in the handout. If you wish you can try this and see what you get.
>
> Students: *(In unison)* Okay Miss. *Students continue to work in their groups on the activity.*

After working on the activity for about 10 more minutes. Ms. M asks students to show her their final solution. Figure 3.29 shows the final proof of group 5 which used Wendy's triangles shown in Figure 3.28.

Ms. M walks around to the groups to observe their solutions. She stops at group five again to look at their solution. She asks the class to look at the group's solutions. Wendy volunteers to explain the solution while writing on the chalkboard.

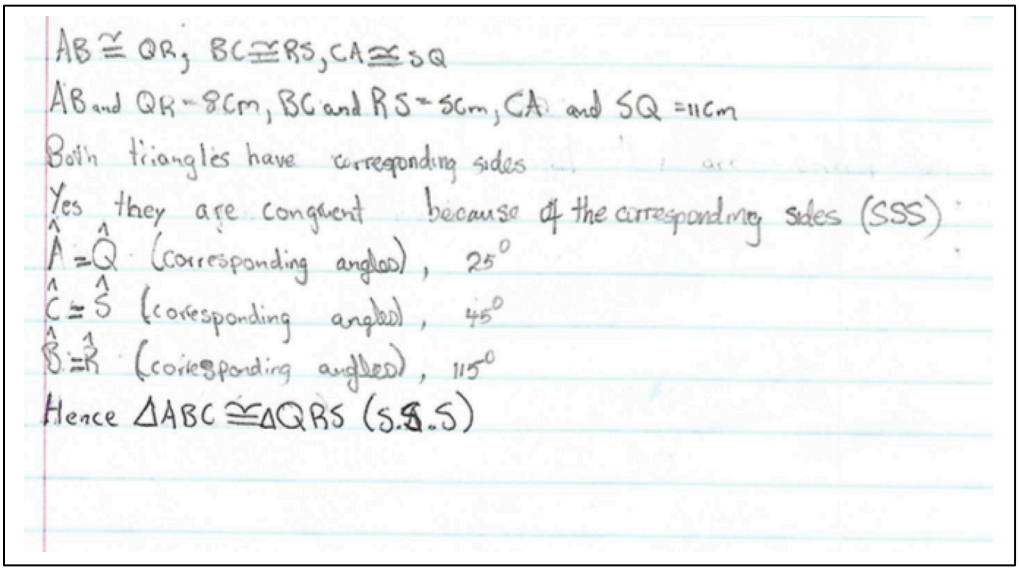


Figure 3.29: Student's proof of congruency using Side-Side-Side postulate

Wendy: Well in our solution we see to answer question 1 in the activity, we found the measures of the three angles in triangle 1 which we labeled *ABC* and then for triangle 2 *QRS*. The angles were $m\measuredangle A = m\measuredangle Q = 25^0$.

Ms. M: Good. Go ahead. What about the other angles?

Wendy: Now $m\measuredangle B = m\measuredangle R = 115^0$ and $m\measuredangle C = m\measuredangle S = 45^0$.

Ms. M: So what does this tell you?

Wendy: Well this tells us that the corresponding angles are equal so we have the triangles are congruent.

Ms. M: Is that the reason why they are congruent?

Ruth: Oh no remember AAA cannot be used to prove congruency so that is not the reason!

Ms. M: What do you think Wendy?

Wendy: Oh yes she is right we cannot use that as a reason for congruency. It is the sides!

Ms. M: What about the sides?

Wendy pauses, Kelly whispers to Wendy and then raises her hand. Ms. M nods at her and she begins to speak.

Kelly: Well we found that the sides *AB* = *QR*, *BC* = *RS* and *CA* = *SQ*.

Ms. M: Okay what does this tell us?

Kelly and Wendy: The corresponding sides are equal!

Ruth raises her hand.

Ruth: Miss remember we have to say because of SSS the triangles are congruent like in the parentheses in the last line of the proof we wrote *(see Figure 3.29)*

Ms. M: Yes that's it.

Wendy: Yes that's what we have to use in this case to prove congruency.

Ms. M: Good job girls, does everyone agree?

Students: Yes Miss

The bell rings signaling the end of the session.

Ms. M explained in the post- interview that this segment of the lesson went very well. Ms. M explained,

Ms. M: I really valued how the segment went, because the students were able to respond to the questions I asked. I did not want to just give them the answers but let them come up with it based on my questions. I also really appreciated how Ruth reminded the class that we agreed that AAA couldn't be used to prove congruence. This shows they are really listening and paying attention.

The bell rings again and students begin to arrange the furniture.

Ms. M announces, "Okay class remember there is a test coming up so please review the work and pass up your group's solution for grading."

Teachers' Pedagogical Decisions

As I explained earlier, this study entails my examination of teachers' pedagogical decisions surrounding the instruction of reasoning and proof. Teachers' pedagogical decisions include, (a) the choice of mathematical tasks, (b) processes of creating opportunities to learn, (c) the instructional strategies teachers use (question, revoicing, direct instruction, cooperative learning), and (d) teachers expectations for students (Martin et al., 2005). My analysis of the

teaching observation data suggested that the aforementioned elements of teachers' pedagogical decisions. In the following sections, I discuss these findings. I provide evidence of the teachers' pedagogical decisions in the previous descriptions of each teacher's instruction. However, in cases where substantial evidence occurred during the lessons I did not describe, I provide brief descriptions of the teaching episode. These descriptions will help exemplify the contextual background of the area of interest I describe in the classroom microculture.

Choice of mathematical tasks

In this study the teachers used three main types of instructional tasks for their students. These were (a) group work, (b) interactive hands-on activities, and (c) problem-based instruction. Table 3.6 summarizes the choice of mathematical tasks and teachers expectations of students.

Table 3.6

Choice of Mathematical tasks for Reasoning and Proof

	Instructional Strategies			Expectation of Students		
Pseudonym	Group work	Interactive Hands-on Activity	Problem Based Learning	Explain Mathematical Thinking	Source of Mathematical Ideas	Responsible for Learning
Ms. V	✓	✓	✓	✓	✓	✓
Ms. M	✓	✓		✓	✓	✓
Mr. P	✓	✓		✓	✓	✓

Note: ✓ = A teachers' choice of mathematical tasks for instruction and expectation of students

As shown in Table 3.6, all three teachers used group work during their instructions. In my descriptions of Ms. V's teaching, she used group work for some activities. Ms. V also used problem-based learning and interactive hands-on activities within these activities. With regard to group work, Ms. V assigned group work activities throughout her teaching during my observations. Ms. V used group work as a means to generate students' ideas and motivate her

students to discover patterns, create definitions, and make conjectures. For example, during the first lesson, Ms. V used group work during the activity entitled, "Congruency in Triangles: Finding your Twin." In this activity, students were required to work in groups of two or three. Ms. V gave each group a numbered card containing measurements of a triangle. There were about 8 different sets of measurements. Therefore at least two groups got the same set of measurements. Ms. V requested that students use their ruler and protractor to draw the triangle from their group's assigned measurements. Students were also required to cut out the triangle they constructed.

After all groups completed their activity. Ms. V asked the students to use their measurements and the cut out of their triangle to find the twin of their triangle, meaning the triangle that is congruent to the one they constructed. One group volunteered and asked, "Who has card number 2?" No one responded and Ms. V continued probing students to help the various groups discover different strategies to find the other group, which had their twin triangle. During the pre-observation interview, in response to my question, "What strategies are you implementing to achieve these goals? Ms. V explained " I would be using group work to help the girls work together to get a solution for finding your twin meaning a pair of congruent triangles." When I probed further by asking "What do you expect to achieve by using group work in your lesson?" she explained,

> Ms. V: It is a good opportunity for students to receive feedback on their ideas within a smaller group before sharing with the entire class. Group work provides a safe space for the shy and even weak students because they may feel more comfortable sharing their ideas and receiving feedback rather than in front the entire class. I also feel that with the weaker students the stronger ones can motivate and guide them.

Ms. V explained her reasons for using group work during her teaching. Ms. V considered group work as a means whereby students can gain the opinion of their peers within a smaller network

of learners. When Ms. V stated, "group work provides a safe space for the shy and even weak students because they may feel more comfortable sharing their ideas," she expressed the view that group work affords all students the opportunity to externalize their thoughts, including possible wrong procedures or answers. Ms. V stated that this was advantageous for the "shy" students or those of a lower mathematical ability. Overall, Ms. V suggested that group work provides a smaller learning space where students can openly share their mathematical ideas.

Additionally, when Ms. V stated, "I also feel that with the weaker students the stronger ones can motivate and guide them," she seemed to indicate evidence of a social constructivist approach to teaching and learning. Ms. V allowed other students to have the role of the more knowledgeable other, who can guide their group members in discussing and refining their mathematical ideas. Ms. V admitted during the post-observation interview that "the group work always make me nervous at times because you become dependent on students being engaged with the activity or appealing to their interests." Ms. V's latter statement suggested her understanding of the possible disadvantages to using group work during her teaching.

Ms. V also used group work along with problem-based learning in the lesson I described in the earlier section. In this lesson, Ms. V assigned the group work during the problem-based lesson "Slow Joe." The group activity helped students make hypotheses about the minimum conditions for congruency. These hypotheses were solutions to solving the problem of "Slow Joe." As I described earlier Ms. V used the real-world problem of a construction worker named Slow Joe. Ms. V used the context of the problem of Joe's slow procedure of measuring all six measurements of a pair of triangle when building congruent roofing trusses. Ms. V explained that Joe's method was time consuming. Students were required to find an efficient method for Joe to construct congruent triangular trusses. Ms. V explained the context of the lesson and provided

the necessary groundwork that supported students reasoning when solving the problem. The goal of the problem was to acquire the minimum necessary conditions to ensure congruent triangles. These conditions were the well-known congruency theorems. Ms. V coupled the problem-based learning with group work to help students engage in the processes of reasoning and proof. In particular students engaged in making conjectures (i.e., their proposed hypothesis for the necessary condition of congruency), which they tested with measuring and deductive reasoning.

Ms. V seemed to use the group work as a means of promoting collaborative learning among her students. Ms. V allowed the groups to present an individual solution for reporting during the whole class discussion. Although she labeled each reporting group's hypothesis after the student presenting, the solution was taken as representative of the mathematical thinking of the entire group. For example, one group's presenter, Cece, reported that her group came up with the Angle-Side-Angle (ASA) condition for congruency. When Cece stated, "Well Miss our group used the measurements *BC,* $m\measuredangle B$, and $m\measuredangle C$," it suggested that the reported solution was taken-as-shared among her group members. Cece's statement indicated that her group collaborated, discussed, and formulated their final solution.

Overall, Ms. V used group work to help students create their own definitions of congruency in lesson one. In her second lesson, group work was used in conjunction with problem based learning to make and test conjectures about the conditions for congruency. In the third lesson, Ms. V gave students opportunities to write proof arguments for questions in the CSEC exams on congruency of triangles. Ms. V firstly demonstrated the correct method of writing proofs that met the examiners' criteria.

After showing two demonstrations for the exam proof questions, Ms. V allowed students to do a hands-on interactive activity from http://illuminations.nctm.org/Activity.aspx?id=3504.

Figure 3.30 shows the hands-on activity as projected on the smart board during the lesson. In this activity, students used the touch screen operation on the smart board to select three triangle elements (i.e., combinations of sides and angles) to prove congruence. The goal of the activity was to test the conditions students hypothesized in the Slow Joe problem during the second lesson. Students used the tools on the screen to construct two triangles with the three elements they used for their hypothesis. For example, if a student selected side *AB*, angle *A* and side *AC* then they will be working with the congruency postulate Side-Angle-Side. The three elements would appear on the screen. Students were required to use the click and drag tool to move the elements to construct a triangle *ABC*. Once the student completed the triangle with the original elements, the triangle moved to the bottom right corner of the screen. A new set of congruent elements would appear. Student used the new set of elements to construct a second triangle. The student then dragged or flipped the second triangle onto the first triangle, and the program indicated whether the triangles are congruent or not. Then the student could conclude that their selected conditions are valid for proving congruence. During the third lesson, Ms. V only used group work at the end of the hands-on activity. Ms. V used group work for students to work on a work sheet on proving congruency. The worksheet consisted of past CSEC examination questions for students to practice their proof writing.

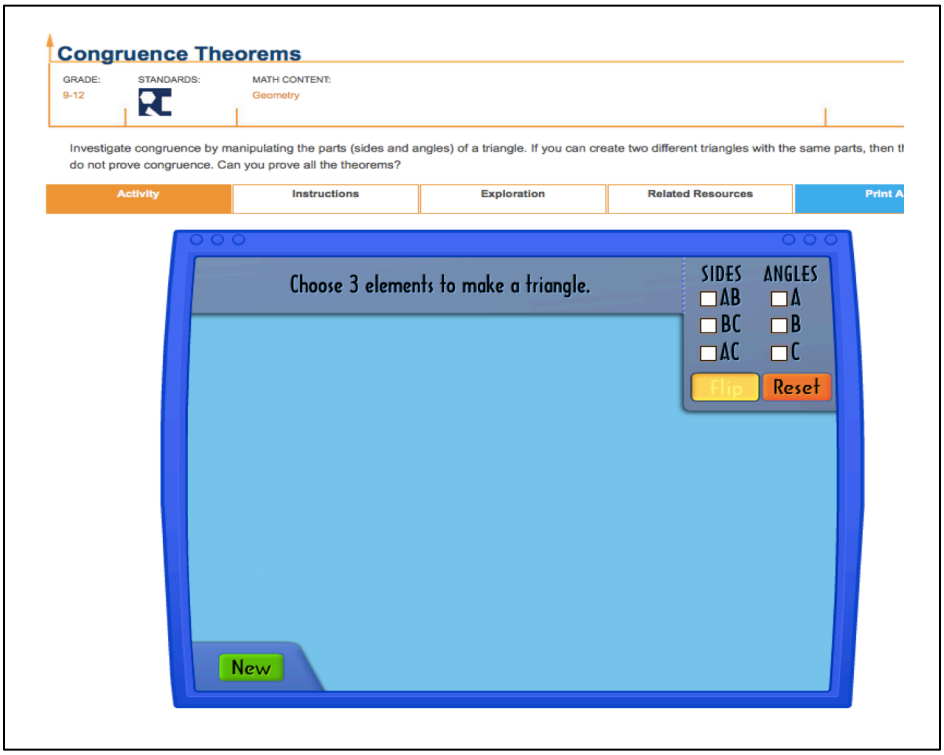


Figure 3.30: Interactive hands-on activity for proving congruence.

Mr. P's Instructional Strategies

Mr. P used group work for students to initially discuss the content before engaging in whole class discussions. Throughout my observations of the lessons, Mr. P assigned various tasks that afforded opportunities for students to create their own definitions of congruency and discovering the postulates for congruence of triangles. During the pre-observation, when I asked the question, "What strategies will you be using to teach the lesson today?" Mr. P explained,

> Mr. P: Well usually in all my classes I have the students assigned to groups so they expect the class to consist of group work. I usually would introduce the topic and then allow the students to work on the group activity and they we will discuss their findings and ideas as a whole class.

Mr. P asserted that group work was his usual choice of mathematical tasks during his teaching. Mr. P stated that he would usually provide the necessary introductory content before allowing students to work on the assigned group activity. Mr. P also suggested that after the introductions, students expected a group work activity that served as a precursor for later whole class discussions. Mr. P expected students to discuss their group's mathematical ideas during the

whole class discussions. This suggested the shift of the mathematical learning community from a smaller group to a larger group of learner (i.e., the entire class). When I probed further by asking "Can you tell me about any activities that engage your students in doing reasoning and proofs?" Mr. P explained,

> Mr. P: I use the group activity to help them come up with ideas together, because by working together they may discover some interesting results that can be a catalyst for further discussions with the entire class or discovery of some new interesting idea later on.

When Mr. P stated, "by working together they may discover some interesting results" he seemed to suggest that the group work forms the basis for students to collectively investigate ideas that lead to the discovery of other mathematical ideas. Additionally Mr. P asserted that the resulting ideas from group activities could potentially be a "catalyst" for whole class discussions and learning of new ideas. This latter comment suggests that Mr. P considered the group work as major vehicle for classroom discourse and extends learning from a group of students to a larger community of learners (i.e., the entire class). For example, during the activity of identifying congruent triangles, Mr. P provided each group with five triangles. He asked students to use their measuring instruments such as a protractor and ruler to measure the triangles and then select the congruent triangles. The students were required to explain how they determined the congruent triangles. As I described earlier in Mr. P's description of this lesson, he solicited students to present their findings so that the entire class can benefit from their ideas. This extended the groups ideas from the group members to the entire class.

Additionally, in agreement with Ms. V's expectation of the group work helping the students guide each other's development of ideas, Mr. P held a similar expectation for group work. When I asked, "In your opinion, what are the most effective ways that teachers can positively influence their students' reasoning and proving skills in mathematics?" Mr. P stated,

Mr. P: Well through group work, students can work together to help each other discover patterns and make conjectures…like those kinds of activities and prove also.

When Mr. P stated, "Through group work, students can work together to help each other" he seemed to consider that during the assigned group activities, some students may have the role as the more knowledgeable other who guides his or her group members to develop their mathematical ideas. Mr. P seemed to hold the view that students are responsible for each other's learning about reasoning and proof. For example, the following excerpt demonstrates the role of students in helping and supporting their group members' mathematical ideas during the lesson. In the excerpt, Randy is reporting his group's solution to a question about proving the congruency of two triangles, *ABC* and *ADC,* which are enclosed in a quadrilateral *ABCD*.

Mr. P: Now everyone remember when someone from a group is reporting you must not talk but listen to their explanations. If you have a question raise your hand and direct it to the person but you must all be quiet and listening.

When the students become completely quiet he asks Randy to proceed with his explanation.

Randy: The two triangles *ABC* and *ADC* are congruent because…Well from the diagram we see that *AB* = *AD* because their measurement is 10. *BC* = *DC* because they are both 10.

Mr. P: So what does this mean?

Randy: Well this means that two of the sides are corresponding.

Mr. P: Good. So Randy has said that the triangles are congruent because the side *AB* corresponds with *AD and BC* corresponds with *DC*. Ok what else will make them congruent?

Randy pauses, he looks at the members of his group, he nods at Penny. Penny smiles, stands, and begins to speak as Randy sits.

Penny: the diagonal $\overline{CA}$ is the same for both triangles because it is like the line of symmetry in the quadrilateral.

Mr. P: The diagonal $\overline{CA}$? What do you mean?

Penny: yes what I mean is that the side *CA* is the same in triangles *ABC* and *ADC*, so because it is the same, all the sides of triangle *ABC* correspond with those of triangle *ADC*. They are equal! Therefore we can conclude they are congruent!

Mr. P: Okay so this group is saying that because all the sides of the triangle *ABC* correspond with and are equal to the sides in *ADC*, the triangles are congruent. Does everyone follow this?

In this episode, Mr. P achieved one of his objectives of allowing students to assist each other during their group discussions and more importantly during their presentations of solutions. Randy was presenting his solution for a problem, which investigated the congruency of a triangle. Randy explained that two sides were congruent. However he was unable to explain how the third side was congruent in both triangles. As a result, he sought the assistance of his group members. Penny responded and assisted Randy in explaining how the common side in both triangles $\overline{CA}$, the diagonal of the quadrilateral *ABCD* was the missing statement in Randy's argument. When I asked Mr. P during the post-observation interview, "What do you think went well in the lesson today?" He talked about the above episode along with other instances when group members helped the person reporting to articulate the group's taken-as-shared ideas or solutions. The students' actions and Mr. P's expectation of students during group work suggests components of a social constructivists approach to teaching and learning. In particular, the students' actions promote the notion of the role of the more knowledgeable other or expert molding and guiding students thinking and ideas.

Ms. M's Pedagogical Practices

Ms. M also seemed to also expect similar results from group work in her classroom. When I asked the question, "Can you tell me about any activities that engage your students in doing reasoning and proofs?" Ms. M stated,

Ms. M: Well I usually use a lot of group work firstly to help the students work together when making and testing conjectures. I think when they collaborate the better

> students can help their group members. However it is important that everyone contributes and they come to a consensus on their final solution or result.

In this quote, Ms. M firstly talked about students having the responsibility for each other's learning. For example when she stated, "I think when they collaborate the better students can help their group members," she seemed to hold the view that the high achieving students can act as the knowledgeable other, who guides his or her group members thinking. However Ms. M had the expectation that all members must come to an agreement among themselves before proceeding. For example, the following excerpt supports Ms. M's aforementioned ideas of group work during the teaching of reasoning and proof.

In this excerpt Ms. M was monitoring the progress of the group's working on the activity investigating the postulates for congruence. After visiting the fifth group, one of the members, Darcy asked Ms. M the following question.

> Darcy: Miss do we need to have the triangle in the same orientation as shown in the handout?
>
> Ms. M: Hmm, well what do you think?
>
> Darcy: Well I think that it could be in a different position and we could see if they are congruent.
>
> Ms. M: Okay what does your group think about this?
>
> *Wendy, another member of the group responds*
>
> Wendy: Well miss we all came up with this at the same time because we started to draw the second triangle in a different position.

In this excerpt the group solicited Ms. M's opinion about their idea concerning the orientation of congruent triangles. Darcy asked the question on behalf of the group. Darcy inquired about whether the different orientation of the triangles could possibly be an alternative method for completing the activity. Ms. M responded by redirecting the question to Darcy. As a

result, Darcy provided her explanation, it seemed that she might be the expert or the more knowledgeable other that proposed the idea. When Ms. M asked, "What does your group think about this?" She seemed to refer to her expectation of the group having a consensus about their final solution. Wendy responded and explained how each member of the group worked on the same aspect of the problem at the same time. Additionally Wendy's explanation suggested that members of the group came to an agreement before proceeding. As a result, the group's collaborated effort supported the idea of Ms. M's expectation for collaborative learning. When the group inquired about their proposed idea of the orientation, they extended their learning space by seeking the advice or the opinion of the more knowledgeable other, their teacher.

Summary

All three teachers used group work to promote the discussion of mathematical ideas. In some cases, Ms. M used group work as a safe learning space where students of different mathematical ability can freely articulate their ideas. All three teachers expected group activities to provide an opportunity for the more knowledgeable students to guide their group member's ideas. However, none of the teachers discussed how they assigned the grouping to ensure that this variation of ability would exist. Nonetheless, all three teachers had the expectation that students would be responsible for each other's learning during group work. Therefore, as the instruction in their classroom moved forward, the responsibility of monitory and guiding ideas, transferred from the teacher to the students.

Mr. P and Ms. V used group work as a precursor for whole class discussions. In both classrooms, students were required to report their findings and thoughts during whole class discussions. This requirement supports the notion that teachers considered students as the source of mathematical ideas. By this I refer to the earlier descriptions when Mr. P stated that the

group's ideas should be the catalyst for whole class discussion and the discovery of other ideas. Ms. V also asserted that the group's ideas should generate the necessary discourse that could achieve the objectives of her lesson. On the other hand, Ms. M used the group work as a learning environment where students can responsibly achieve the lessons objectives. Although Ms. M did not use whole class discussions after the group work, she monitored the groups to ensure that their discussions achieved the lesson's objective. Therefore, Ms. M allowed the students to be responsible for their learning and fostered opportunities for students' collaborative learning.

The Role of Hands-On Activity in The Instruction of Reasoning and Proof

Mr. P used a hands-on activity during his instruction. During the first lesson Mr. P asked students to categorize several shapes on the white board (see Figure 3.16). Students volunteered and classified the cut out shapes into groups based on color, shape, and size. Mr. P used whole class discussions to help students define congruency based on the results of their classification. In this case, the hands-on activity acted as a precursor to defining congruency. Ms. V used a hands-on activity to further test students' conjectures about the necessary conditions for congruency. As a result of this activity, students confirmed the validity of their conditions. Students were now motivated to use these valid conditions as congruency postulates for proving congruence. The activity served as a precursor for proof construction.

Instructional Strategies of Teachers for Reasoning and Proof

Table 3.7 shows the various instructional strategies evident in the teachers' instruction of congruence. All three teachers used questioning and revoicing during their lessons. By questioning, I mean teachers' use of open-ended questions that allow students to explain their mathematical thinking. Teachers used of open-ended questions to present an open-ended task to their students. The task would require (a) further explanation of one's thinking, (b) clarification

of one's ideas, (c) providing additional information, (d) requests for conclusions or next steps in a reasoned argument, or (e) requests for evaluations of others' arguments.

By revoicing, I refer to the teacher's action of (a) repeating or rephrasing students' comments or questions, (b) requesting further explanation, or (c) emphasizing key ideas in students' statements or thinking (Martin et al., 2005). During the lesson, each teacher demonstrated aspects of revoicing during whole class discussions. In the next section, I discuss the teacher's instructional strategies evident in their teaching of reasoning and proof.

Table 3.7

Instructional Strategies of Teachers for Reasoning and Proof

Pseudonym	Questioning	Re-voicing	Direct Instruction
Ms. V	✓	✓	✓
Ms. M	✓	✓	
Mr. P	✓	✓	

Note: ✓ = A teacher used this instructional strategy during reasoning and proof lesson

Example of Mr. P's Instructional Strategies

In this episode Mr. P solicited solutions to the question in Figure 3.31 below. The question required that students determine the congruent triangles in the given geometrical object. Students had to identify which triangles in the pyramid shaped object were congruent. One of the students volunteered and began to explain her group's solution.

\

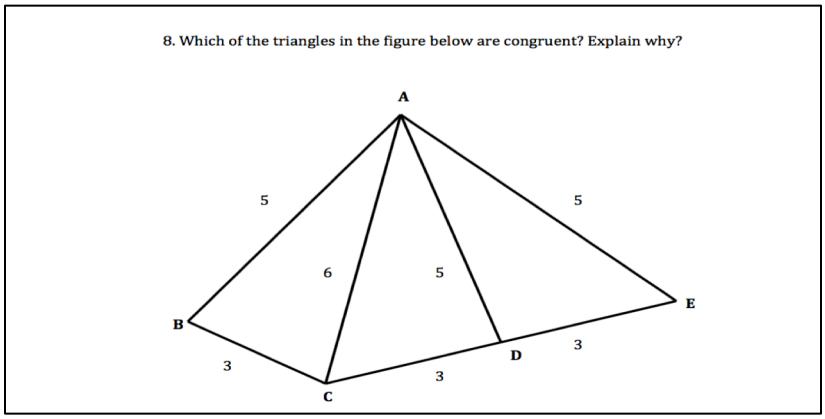


Figure 3.31: Mr. P's assessment question on congruent triangles

Delina: Well we found that triangles *ABC* and *ACD* are congruent because *AC* is the same in both triangles and *AB* = *AD* = *5* and *BC* = *CD* = *3*. Therefore by SSS we can say that triangles *ABC* and *ACD* are congruent.

Mr. P: Okay good! Delina is saying that from her group, triangles *ABC* and *ACD* are congruent because of three corresponding equal sides. Okay what about *ADE*?

Delina: Well the first two triangles are congruent but the second and third are not because all their sides are not the same.

Mr. P: Okay which ones?

Delina: Well *AB* = *AD* and *BC* = *DE* but *AC* = *6* and is not equal to *AD* which is *5*.

Mr. P: Yes good job. Does everyone agree with this explanation and answer?

Students: *(In unison)* Yes Sir.

Mr. P: Okay, let us move on a bit because the bell will ring soon.

In this episode, Delina shared her group's solution for the question. Delina indicated that triangles *ABC* and *ACD* were congruent. Delina also fulfilled the expectation of providing explanations of why the triangles were congruent. Delina provided the supporting reason that the three corresponding sides in the two triangles were equal. As a result, she concluded that by the SSS postulate, the triangles were congruent. When Mr. P stated, "Ok good" he evaluated Delina's response. Mr. P also indicated his agreement with Delina group's solution. Once Mr. P evaluated the response, he revoiced Delina's explanation for congruence of the two triangles. Mr.

P summarized and rephrased Delina's statements. Additionally, when he stated, "Delina is saying from her group", he attributed ownership to the student presenting. Mr. P's action also demonstrated characteristics of reform-based teaching (Hufferd-Ackles et al., 2004), that is the student or group was the source of the mathematical ideas being presented to the class.

After revoicing, Mr. P went further and posed an open-ended question, "What about *ADE?*" Mr. P's question solicited an explanation for the status of the other triangle in the given diagram. Mr. P's inquiry was directed to Delina and her group. Mr. P expected an explanation about the congruency of the triangle *ADE*. Delina's response seemed to be directly related to the initial question "Which of the triangles in the figure below are congruent?" Delina relabeled the triangles as first triangle (*ABC)*, second triangle (*ACD)*, and third triangle (*ADE*). Delina used these labels to explain which pair triangles in the figure were congruent. Delina also explained that triangle *ADE* was not congruent to the second triangle *ACD* thus possibly implying by hypothetical syllogism, that *ADE* was also not congruent to *ABC*.

In her response to Mr. P's expectation of explaining one's mathematical thinking, Delina provided a reason explaining why the second and third triangles were not congruent. Delina explained, "Because all their sides are not the same." Mr. P seemed to require further clarification of her answer so he posed another question, "Which ones?" This open-ended question motivated Delina to provide further clarifications about which corresponding sides in triangles *ADE* and *ACD* that were not equal. After Delina provided her answer, Mr. P evaluated her response by saying, "Yes, good job." Mr. P then extended the evaluation by allowing her peers to evaluate her response by saying, "Does everyone agree with this explanation and answer?" This action demonstrated a shift in the responsibility of learning. By this I mean that Mr. P moved the evaluation of Delina's explanation and answer to the other students in the class.

Mr. P opened up the discussion so that students can reach a consensus and students also had the opportunity to provide feedback on Delina's response. Based on the class's response, it seemed that Delina's answer met the requirements of what is an acceptable answer in Mr. P's classroom. Table 3.8 summarizes Mr. P's instructional strategies and the expectations of students in the aforementioned excerpt of his instruction.

Table 3.8

Summary of Mr. P's Instructional Strategies

Examples of Teacher's Instructional Strategy	Expected Tasks of Students
Solicit a solution for the problem.	Provide a report of group's solution.
Evaluate, revoice, and pose open-ended questions: Okay what about *ADE*?	To provide further explanation about the other triangle in the figure and draw a conclusion.
Evaluate and pose open-ended question: which ones?	Provide further clarification of the equality or non-equality of corresponding sides.
Evaluate, ascertain consensus among students and extend evaluation of the student's solution to the entire class.	Provide feedback and confirm agreement with presented solution.

Example of Ms. V's Instructional Strategies

Ms. V used questioning, revoicing, and direct instruction during her teaching. In the following episode Ms. V discussed the changing of the notation of a student's suggested condition for proving congruence. Two of the groups suggested using two sides and the enclosed angle as measurements Joe could use to solve his problem of obtaining congruent triangles. In their responses they listed the two sides and then the angle. As a result, Ms. V denoted their suggestions as Side-Side-Angle (SSA). However, one of the students who reported on behalf of her group suggested a change in notation. The following dialogue took place,

Tia: Miss because the order is important we should probably change the notation to Side Angle Side.

Ms. V: Okay we have a suggestion from Tia

Students stop writing and listen to Ms. V.

Ms. V: Now Tia is saying that we should change the notation from SSA to SAS because the order is important. What do you think?

Melissa raises her hand, Ms. M motions to her to speak.

Melissa: Miss I think that is better for us to remember the order if you have SAS.

Tia: That's the reason I think we should change it because the angle must be in the center of the two sides.

Ms. V: Does everyone agree?

Students: Yes Miss.

Nicki waves her left hand while writing in her notebook. Ms. V on seeing her hand says,

Ms. V: Okay Nicki do you have a question?

Nicki: Yes Miss.

Ms. V: Okay class let us listen to Nicky's question.

Nicki: Why should we change it?

Ms. V: Because it indicates the importance of having the angle enclosed by the two sides.

Nicky nods her head and says,

Nicki: Okay I get it.

Ms. V: Do we agree?

In this episode, Tia offered a suggestion for the notation of the condition of two sides and the enclosed angle. Ms. V acknowledged Tia's suggestion and notified the other students of the new proposed idea. Ms. V revoiced Tia's statements. When Ms. V stated, "Tia is saying that we should change the notation from SSA to SAS" she gave credit to Tia as the source of this new

mathematical idea. In a similar manner to Mr. P, this action suggested evidence of one of the characteristics of reform-based teaching (i.e., students are the source of mathematical ideas). When Ms. V asked, "What do you think?" Ms. V opened up the discussion so that the other students could offer their feedback on Tia's suggestion. As a result, Melissa provided feedback. When Melissa stated, "Miss I think it is better for us to remember the order if you have SAS." She suggested her agreement with the solution. Melissa also expressed her view that the SAS acronym emphasizes the order, which in her opinion would be easier to recall. Tia supported Melissa's statement by affirming her reason for changing the notation to SAS. The letter A in the center emphasizes that the angle lies between the two sides.

Ms. V then extends the affirmation by asking, "Does everyone agree?" This question solicited a consensus among all students before proceeding. However, one of the students needed clarification. Nicki gestured by waving her hand to indicate that she wanted to ask a question. When Ms. V asked, "Ok Nicki, do you have a question?" she confirmed the meaning of Nicki's gesture. Ms. V also acknowledged Nicki's role as a questioner. Ms. V notified the class that Nicki needed to ask a question about the new idea. When Nicki asked, "Why should we change it?" Ms. V did not revoice or redirect Nicki's question. Ms. V chose to answer the question directly. However, when Nicki responded that she understood Ms. V's explanation, Ms. V yet again asked a question, which opened up the discussion for the other students' feedback. In this episode Ms. V used revoicing and open-ended question as instructional strategies. Table 3.9 summarizes Ms. V's instructional strategies and expectations of students during this excerpt of her teaching.

Table 3.9

Summary of Ms. V's Instructional Strategies

Example of Teacher's Instructional Strategy	Expected Tasks of Students
Revoice student's suggestion, seek evaluation and consensus from other students, and pose open-ended question: What do you think?	Provide feedback on proposed solution.
Pose open-ended question: Does everyone agree? Ascertain consensus among students and extend evaluation of the student's solution to the entire class.	Provide feedback and confirm agreement with presented solution.
Acknowledge student as questioner: Do you have a question?	Pose question to teacher or other students.
Ascertain consensus among students.	Confirm agreement with presented solution.

Direct-Instruction During the Teaching of Reasoning and Proof

Ms. V used direct instruction during the last lesson on congruence. By direct instruction, I mean the teacher's use of stand-alone recitation to teach content. The teacher does not offer opportunities to ask questions or provide feedback. During the third lesson, Ms. V used direct instruction to demonstrate how to correctly write proofs for questions in the CSEC examination. Figure 3.32 shows the examination question as projected on the smart board. In the question, students are required to use the information provided to determine which postulate for congruence could be aptly used to answer the question correctly. The following excerpt took place.

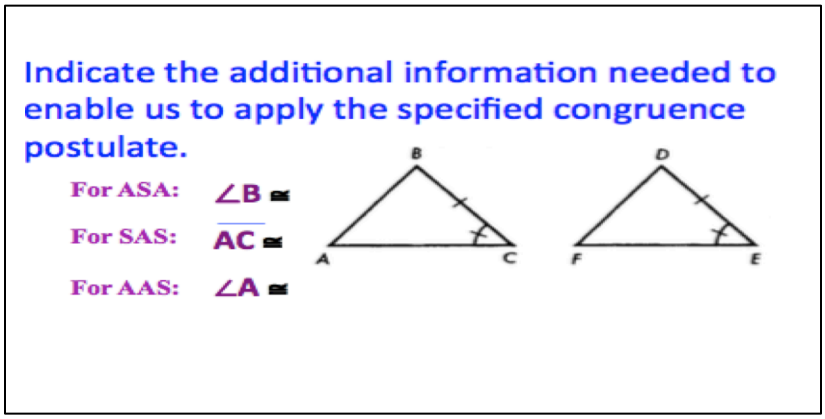


Figure 3.32: An example of a CSEC examination question, Ms. V used during lesson three

Ms. V walks to the smart board and announces, "Okay, we have a question from the CSEC exam. Now the examiners ask us to prove congruence of the given triangles using the information provided in the diagram. Now to do this correctly and to get the most marks from the examiners you need to write the following." Ms. V proceeds to explain the students that for ASA, they will need two angles and the enclosed sides. To use this postulate they will need to identify the corresponding segments thus angle *B* must be congruent to angle *D* and the segment $\overline{AC}$ must be congruent to the line segment $\overline{FE}$ and the angle *A* must be congruent to angle *F*. Ms. V does not solicit any suggestions from the students. She explains each of these statements and then proceeds to show them the solution to this question as expected by the examiners. Figure 3.33 shows the solution as projected on the smart board. Ms. V explains the notations to the students.

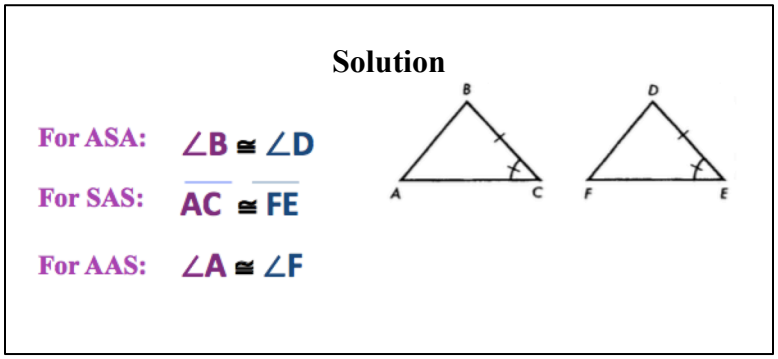


Figure 3:33: Ms. V's solution to a CSEC question

In the post-observation of this lesson, I asked Ms. V, "Is there anything that you would do differently?" She explained,

> Ms. V: Due to time constraints I needed to show the students the correct method for writing the proof arguments for the CSEC examination. I felt that by writing all the statement and explaining why they need the notations, students would be able to understand the nuances of answering the questions accurately. I really would have liked to let them offer suggestions but I needed to move on so they could get to the worksheets.

Ms. V explained how the issue of time constraints affected her intentions of allowing students to provide the solutions to the problem. However, she did indicate her preference for telling them directly the specific notations and requirements of a correct answer. These requirements were the expectations of an answer that would receive the best mark from the examiners. Ms. V direct instruction helped her clarify the examiners expectations and also introduce the required notations.

Example of Ms. M's Instructional Strategies

During the observed lessons, Ms. M used revoicing and open-ended questions during her instruction. However, among the three teachers, Ms. M used open-ended questioning the most and demonstrated various expectations of students. I selected the following excerpt to exemplify the use of open-ended questioning as an instructional strategy. In this teaching episode, Ms. M is reviewing students' solution for the student activity about proving congruence. The question required that student use the SSS postulate to prove that the two triangles they constructed were congruent. Students worked on the activity in groups and one of the groups was reporting their solution. Ms. M walked solicited a group to present their solution. Wendy volunteered and began to explain the solution while writing on the chalkboard. Figure 3.34 shows the group's solution for the activity.

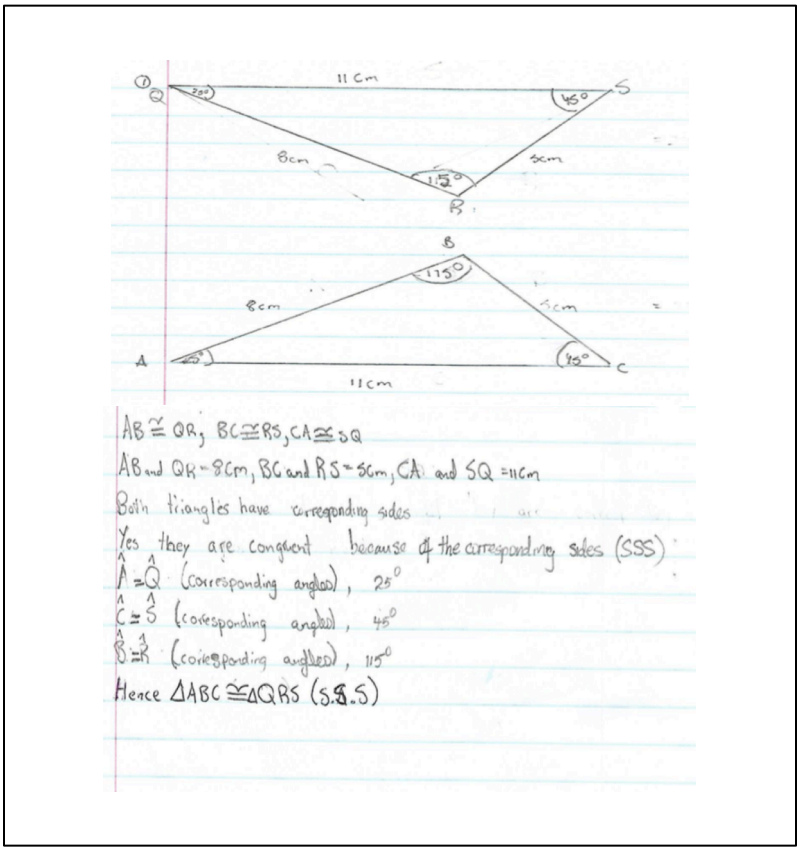


Figure 3.34: Group's solution to problem on proving congruence in Ms. M's class

Wendy: Well in our solution we see to answer question 1 in the activity, we found the measures of the three angles in triangle 1 which we labeled *ABC* and then for triangle 2, *PQR*. The angles were $\mathrm{m}\measuredangle\mathrm{A} = \mathrm{m}\measuredangle\mathrm{Q} = 25^0$.

Ms. M: good go ahead, what about the other angles?

Wendy: Now $\mathrm{m}\measuredangle\mathrm{B} = \mathrm{m}\measuredangle\mathrm{R} = 115^0$ and $\mathrm{m}\measuredangle\mathrm{C} = \mathrm{m}\measuredangle\mathrm{S} = 45^0$.

Ms. M: So what does this tell you?

Wendy: Well this tells us that the corresponding angles are equal so we have the triangles are congruent

Ms. M: Is that the reason why they are congruent?

Ruth: Oh no remember AAA cannot be used to prove congruency so that is not the reason

Ms. M: What do you think Wendy?

Wendy: Oh yes she is right we cannot use that as a reason for congruency. It is the sides.

Ms. M: What about the sides?

Wendy pauses, Kelly whispers to Wendy and then raises her hand. Ms. M nods at her and she begins to speak.

Kelly: Well we found that the sides $AB = QR$, $BC = RS$ and $CA = SQ$.

Ms. M: Okay what does this tell us?

Kelly and Wendy: The corresponding sides are equal!

Wendy raises her hand. Ms. M nods at her

Ruth: Miss remember we have to say because of SSS the triangles are congruent like in the parentheses in the last line of the proof we wrote *(see Figure 3.34)*

Ms. M: Yes that's it!

Wendy: Yes that's what we have to use in this case to prove congruency.

Ms. M: Good job, does everyone agree?

Students: Yes miss.

In this episode, Wendy correctly identified the corresponding angles A and Q was equal. Ms. M first evaluated Wendy's response by saying "good go ahead". The last phrase also suggesting Ms. M's prompt for the student to continue her explanation. The statement also suggested Ms. M's expectation for more statements, in particular, about the other angles. When Ms. M posed the question, "What about the angles?" she presented an open-ended tasks that required further explanations and additional information about Wendy's observations of the other angles in the two triangles. Wendy explained that the other corresponding angles were equal. Ms. M immediately posed another open-ended question "So what does this tell you?" This question prompted Wendy to make a conclusion to complete her argument about the congruence. Wendy used the fact that the corresponding angles were equal (i.e., AAA), to support her conclusion about the congruence of the triangles. Ms. M realizing the error in the conclusion, decided to redirect the evaluation to the other students by asking "Is that the reason

why they are congruent?" This question presented the task that required students' feedback on the incorrect reason for the congruence of the two triangles.

Another student and group member responded and reminded the class that AAA does not prove congruence. Ruth presented the classroom mathematical practice for proving congruence. In earlier discussions, the students and Ms. M agreed that as a mathematical practice the equality of all three corresponding angles (i.e., AAA) does not prove congruence. Therefore this fact could not be used to prove a pair of triangles was congruent. Ruth reminded the students and her group member of this classroom mathematical practice. Wendy agreed with Ruth's reminder and was able to correctly identify that her observations about the sides of the triangles should be used instead to prove congruence.

Ms. M posed an open-ended question, "What about the sides?" This question prompted Wendy's reasoning about the sides of the triangle. Wendy seemed unable to complete the task, and her group member, Kelly, assisted her. Kelly stated the equal measurements for the three corresponding sides. Ms. M evaluated the response and decided to prompt the students further by asking again "What does this tell us?" Ms. M chose to use the open-ended question to allow the students to state the next step in their argument. Ms. M's question seemed to request more information in support of the students' conclusion about the congruence. Eventually, Kelly and Ruth concluded that the corresponding sides were equal. However, Ruth seemed to realize that her group members needed to provide more information. Thus, Ruth introduced another classroom mathematical practice. Ruth reminded the students that the SSS postulate must be used as a reason to support their statements about the corresponding equal sides and their eventual conclusion about congruence. During the earlier discussion in the lesson, Ms. M and her students established the mathematical practice that SSS is an acceptable result for justifying the

congruency of a pair of triangles. Ms. M and Wendy both agreed with Ruth's reminder of the mathematical practice. Ms. M then asked the question, "Does every one agree?" to ensure there was some consensus on the explanation and also to requests for evaluations from the other students if there was not any agreement.

This episode highlighted Ms. M's use of questioning. Table 3.10 summarizes the questions posed and the expectation of the students when responding. Ms. M asked questions that prompted (a) further explanation of students' mathematical thinking, (b) clarification of students' ideas, (c) providing additional information that explains students' observations about equal measurements, (d) conclusions or stating the next steps in a reasoned argument, and (e) requests for evaluations of another students' arguments. In this episode, I provided an example of the teachers' extensive use of questioning during their instruction. Although the selected teaching episode highlights the use of questioning, Ms. M also used revoicing during the lesson.

Table 3.10

Summary of Ms. M's Use of Question as an Instructional Strategy

Types of Teacher's Question	Expected Tasks of Students
What about the other angles?	Provide additional information.
So what does this tell you?	To draw a conclusion or provide the next steps in a reasoned argument.
Is that the reason why they are congruent?	Provide further explanation of one's mathematical thinking.
What about the sides?	Provide additional information.
What does this tell us?	Draw a conclusion or provide the next steps in a reasoned argument.
Does everyone agree?	Evaluate another student's arguments.

Overall the teachers used various instructional strategies and mathematical tasks to promote opportunities for reasoning and proof. Figure 3.35 provides a summary of my previous discussion of the findings associated with the components of teacher' pedagogical decisions. As shown in this figure the * denotes the cases where my analysis suggested evidence of the particular component of teachers pedagogical decisions. In the next section, I discuss the sociomathematical norms evident in the teachers' instruction of reasoning and proof.

Sociomathematical norms

In this section I present the sociomathematical norms the teachers established during their instruction. By sociomathematical norms, I refer to the norms teachers established specifically to support their students' mathematical thinking during their instruction of reasoning and proof. In order to encompass the implications of such norms within the reform-based instructional policies, I discuss these norms along the characteristics of reform-based teaching. As an area of interest to the entire dissertation study, I considered Hufferd-Ackles et al.'s (2015) description of characteristics of reform-based instruction. These included questioning, explaining mathematical thinking, sources of mathematical ideas, and the responsibility for learning. I use these four characteristics to guide my presentation of the observed sociomathematical norms in the teachers' instruction.

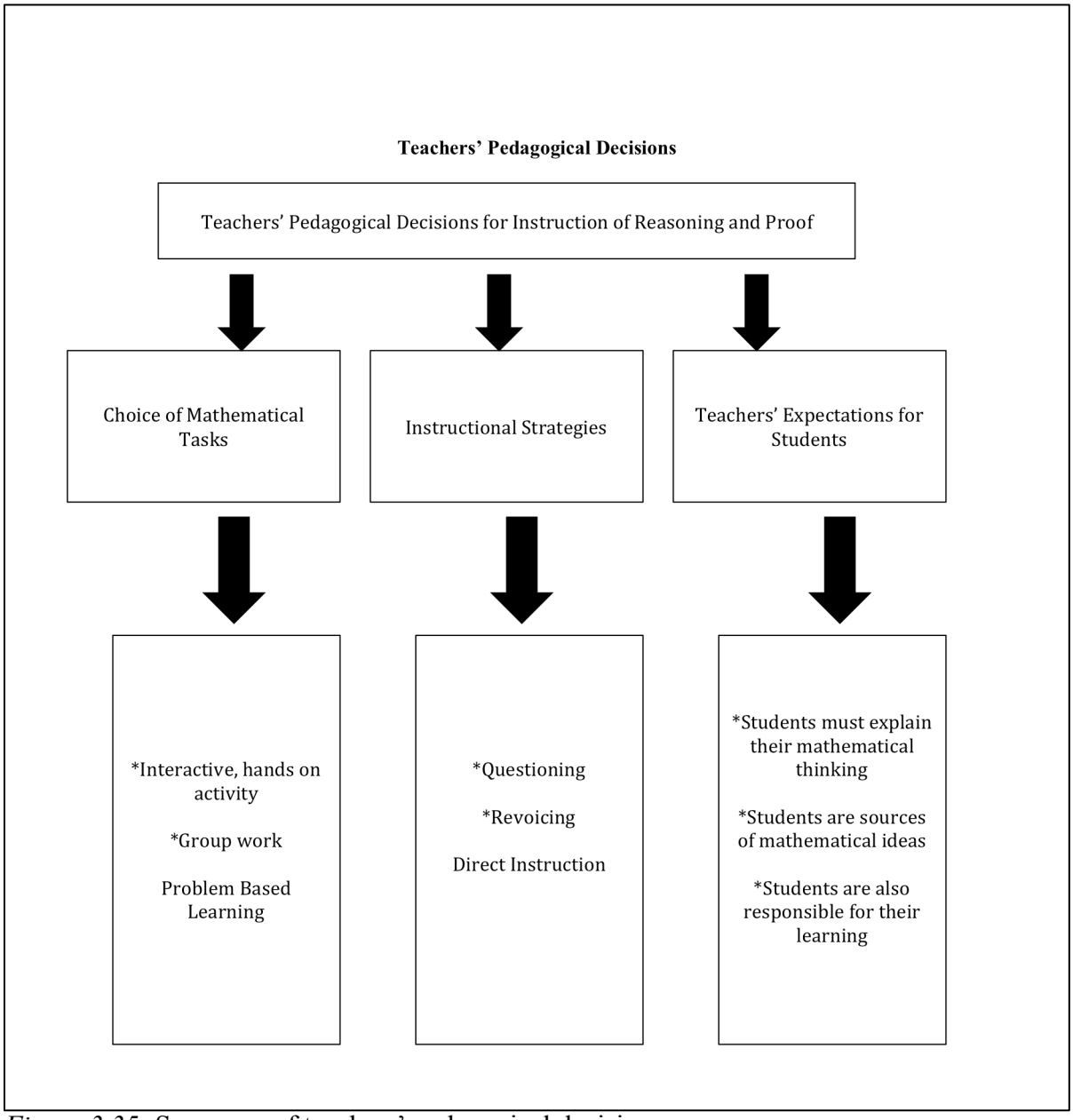


Figure 3.35: Summary of teachers' pedagogical decisions.

Table 3.11 presents a summary of the sociomathematical norms evident in the teachers' instruction. The first column presents the characteristics of reform-based instruction; the second column provides a description of the characteristics. The remaining columns present examples of each of the teacher's sociomathematical norm relative to the characteristic of reform-based

instruction. In the next sections I discuss the sociomathematical norms for each characteristics with supporting evidence from the respective teacher's instruction.

Questioning

Within reform-based teaching, questioning describes the role of students as questioners. Students ask questions that motivate mathematical reasoning, justification of claims, and understanding content. My observations of teaching suggest that during two of the teachers established norms during the observed lessons whereby it was normative for all students to actively pose questions. Students asked questions that either prompted explanations of mathematical thinking and sharing of ideas or sought clarification of an idea. For example, during Ms. M's instruction, it was typical for more than half of the students to comment on each other's ideas and students actively posed questions to their classmates. Therefore these observations suggest that Ms. M established the custom that students can either comment or pose questions during whole class discussions. Ms. M usually reminded students at the start of the class or during the lesson about asking questions. During the start of the lessons Ms. M announced "Ok students we are going to look at congruency and remember if you need to ask a question about something we are discussing please do so." In this announcement Ms. M established that students can act as questioners if needed for clarification of ideas or answers being discussed. The following example shows a case where Ms. M established the sociomathematical norm of posing questions to clarify students understanding of mathematical ideas being presented. In this excerpt Ms. M was discussing the postulates for congruency. One student had previously explained the Angle-Angle-Side postulate. Another student posed a question that led to the clarification of the difference between AAS and ASA postulates for congruence.

Ms. M writes the acronym AAS with its meaning on the chalkboard. Students continue to write in their books. Morgan raises her hand Ms. M points to her,

Ms. M: Do you have a question?

Morgan: Yes I have a question about the last one you wrote on the chalkboard.

Some students begin to talk to each other. Ms. M says, "Quiet everyone! Let us listen to Morgan's question, remember we need to allow everyone to either comment or ask a question and you must be listening and paying attention." The class becomes quiet and Morgan begins to speak,

Morgan: Well doesn't it have something like ASA also where the S is in the middle?

Ms. M: So Morgan is asking if there exists an ASA postulate? Does anyone have an answer for this?"

In the above excerpt Morgan wanted to ask a question about the AAS postulate. Ms. M confirmed that Morgan needed to ask a question. Ms. M realizing the need to have Morgan direct her question to the entire class, called the class to order. Ms. M also seemed to adhere to her earlier established norm about students having the right to be questioners. Ms. M reminds the class of the norm about students having the right to ask questions. In particular when she stated "Let us listen to Morgan's question, remember we need to allow everyone to either comment or ask a question and you must be listening and paying attention." This latter phrase emphasized the responsibility of all students as either a questioner or listener. In both roles students adhered to the sociomathematical norm about questioning during the lesson.

Table 3.11

Sociomathematical Norms Evident in Teachers' Instruction of Reasoning and Proof

Category of Reform-based Teaching	Description	Examples of Sociomathematical Norms		
		Ms. V	Ms. M	Mr. P
Questioning	Students ask each other questions that motivate mathematical reasoning, justification, and understanding		Ms. M established the norm that students could ask questions during the lesson.	Mr. P established the norm that students can ask questions to another student after or during a student presentation by raising their hand.
Explaining mathematical thinking	Students explain their solutions using mathematical argumentation.		Ms. M encouraged students to explain their thinking, a acceptable answer must include an explanation of one's mathematical thinking processes.	Mr. P established that an acceptable proof must have an explanation of why the conclusion holds for congruency.
Sources of Mathematical Ideas	Students can compare their strategies looking for mathematically important similarities and differences.	Ms. V established that a different solution could be (a) the same answer but a different approach. For example, when students came up with different approaches for SAS as their solution to "Joe's Problem" during lesson 2.		Mr. P established the norm that all answers are acceptable but must go through the feedback within the group or in whole class discussion. If it has an error the class can discuss and fix the error.

	Students use mistakes as an opportunity to rethink their conceptions of mathematical ideas and as a catalyst for seeing new ideas.	(b) A different answer to the same problem. E.g., When students came up with ASA and AAS as different solutions to "Joe's Problem "during lesson 2. In both cases the solution must have supporting reasons why this approach or solution is valid.		For example, when a group presented AAA as a congruency postulate. Mr. P used this suggestion as an opportunity to show a counterexample of AAA and the class discussed why it would not work.
Collaborative Learning/ Responsibility for Learning	Students must reach a consensus using mathematical reasoning and proof during group work. Students are responsible for learning.	Ms. M established the norm that after revoicing or rephrasing a student's solution or question, students are expected to provide feedback or evaluation of the proposed idea.	Ms. M established the norm that students must give each other feedback and collaborate their final solution when doing group work	Mr. P established the norm that the entire group must always agree on their solution before presenting to the class.

Mr. P established the norm that students can pose questions to other students after or during their presentation. Mr. P stated during his announcements before groups reported their findings "Ok students, stop writing, finish your discussions and sit up. Remember if you have a question while a group is reporting, you can raise your hand and ask your question." In this quote Mr. P established the procedure for a student to pose a question to another student. Mr. P's announcement established that it was a normative custom during his instruction for students to act as the questioners.

Explaining Mathematical Thinking

Teachers established sociomathematical norms that required students to explaining their mathematical ideas. Ms. M established the norm that motivated her students to explain their thinking. This sociomathematical norm established what is an acceptable answer. According to Ms. M, an acceptable answer must include an explanation of one's mathematical thinking processes or supporting reasons for one's claims. For example, the following excerpt presents the case where Ms. M demanded an accompanying explanation with students' solutions or responses to her questions.

Angle-Angle-Side Episode

After writing the previous postulate on the chalkboard, Ms. M walks around the room as students continue writing in their notebooks. She then walks to the front of the class room again and asks the question "Can someone give me another one?" Dale raises his hands and answers and says "AAS."

Ms. M nods and writes on the chalkboard the acronym AAS and then asks,

Ms. M: What does that mean? Remember class you need to explain!

Dale: Well it means Angle-Angle-Side and this is when you have two corresponding angles are equal and one side also equal.

Ms. M: So Dale is saying that AAS means two angles are equal and one side is also equal.

In this excerpt Ms. M solicited an answer about the congruency postulates. Ms. M posed the question "Can someone give me another one?" Dale responded and suggested AAS. When Ms. M asked Dale "What does that mean?" she indicated the need for Dale to fulfill his obligation of providing an acceptable answer. Ms. M went further and reminded the students of the established norm of what counts as an acceptable answer in her class. In particular, when she stated, "Remember class you need to explain!" she suggested that an acceptable answer must have an explanation of one's reasoning or supporting arguments.

As an expectation of students explaining their mathematical thinking, Mr. P established that a valid proof must have statements and supporting reasons, which explain why the conclusion holds. In particular, during Mr. P's third lesson, he discussed with his students, how to write a proof for a CSEC examination question. Mr. P engaged students in a discussion about what steps should be placed in answering the question shown below in Figure 3.36. The question required that students prove that triangles *ABE* and *DCE* are congruent given that the line segments $\overline{AB}$ and $\overline{CD}$ are parallel. Mr. P reminded students during the following discussion about the sociomathematical, which describes what counts as a valid proof argument. The following demonstrates Mr. P's expectations during the discussion of the CSEC question.

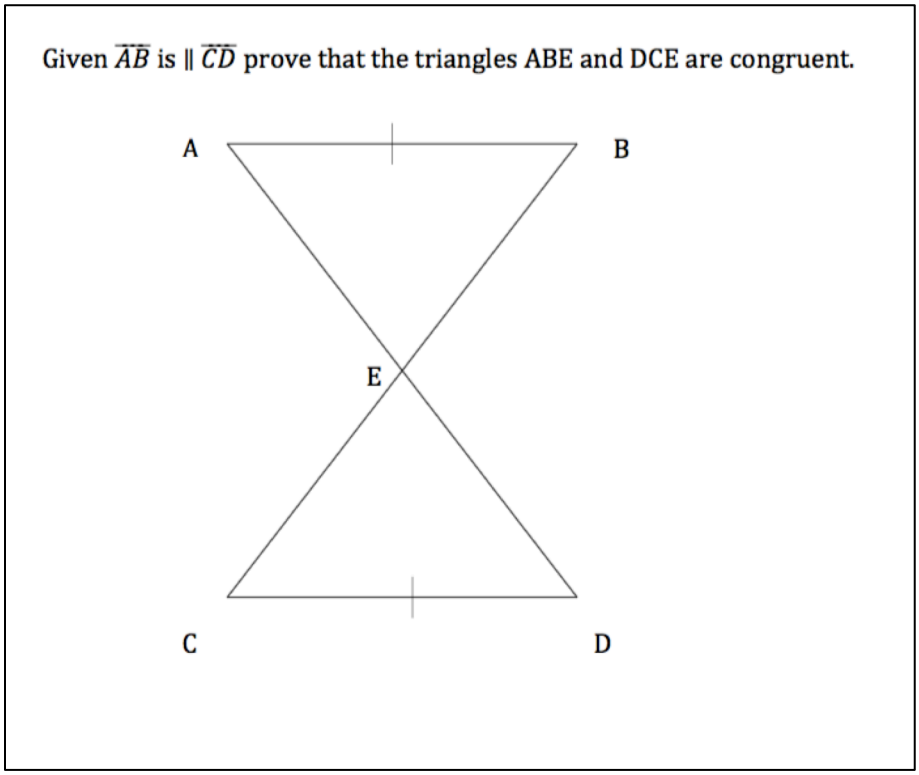


Figure 3.36: Example of a CSEC question Mr. P during lesson 3.

Mr. P: Now to prove this what is some of the things we can claim?

Shane: Well Sir using the sides *AB* and *CD* we know they are corresponding and equal.

Mr. P: Why Shane?

Shane: Well because of the markings in the given diagram we know from the given information they are equal and they also told us they are parallel.

Mr. P: Okay good. What else? Someone else continue.

Ava: Well Sir as Shane said they are parallel and the angles *B* and *C* are corresponding and equal.

Mr. P: Okay Why are they equal?

Ava: Well because of the Z angles that are formed by the transversal $\overline{BC}$ cutting across the parallel lines $\overline{AB}$ and $\overline{CD}$.

Mr. P: Okay good so Ava is saying because of the alternating angles or Z angles formed by the transversal cutting across the parallel lines. Could someone else continue?

Randy: Well the angles A and D are also equal.

Mr. P: Okay Why? Now class remember you need to not just write statements like angles B and C are equal. You must provide the reason because they are alternating or Z angles formed when the transversal crosses the parallel lines. Okay! Do we agree on this?"

Students: Yes Sir.

Mr. P: Okay so Randy what must you add to this proof

Randy: Well I needed to say that the angles A and D are also equal because of the Z angles that are formed by the transversal $\overline{AD}$ cutting across the parallel lines $\overline{AB}$ and $\overline{CD}$.

Mr. P: And what do you conclude?

Randy: Well because $m\sphericalangle A = m\sphericalangle D, AB = DC$, and $m\sphericalangle B = m\sphericalangle C$ we have by ASA postulate, and $\triangle ABE \cong \triangle DCE$.

Mr. P: Good job! So remember your proof must have the statements and their reasons else you will not get all the marks.

In the above excerpt Mr. P requested suggestions for proving that the triangles *ABE* and *DCE* were congruent. Shane and Ava offered possible claims that could be used in the proof. However, in both instances Mr. P prompted them for the supporting reasons for the validity of their claims. When the third student, Randy offered his suggestion without a reason, Mr. P reminded the students of the sociomathematical norm about what counts, as an acceptable proof. When Mr. P stated, "Now class remember you need to not just write statements like angles *B* and *C* are equal. You must provide the reason" he established what counts as a valid proof in his classroom. Mr. P also ensured that students also agreed with the sociomathematical norm he established by asking, "Do we agree on this?" Mr. P's action of establishing this sociomathematical norm prompted Randy to correctly state his claims with supporting reasons. Therefore the students were able to construct an acceptable proof argument.

Sources of Mathematical Ideas

With regard to students as the source of mathematical ideas, Ms. M established a norm that defined what counts as a different solution during class discussions. This norm was important since it allowed students to compare their strategies by looking for mathematically important similarities and differences. Ms. V established that a different solution could be the same answer but a different approach. For example, when students came up with different approaches for the SAS condition as their solution to "Joe's Problem" during Lesson 2. Ms. V also established that a different solution could be a different answer to the same problem. For example, when students came up with Angle-Side-Angle (ASA) and Angle-Angle-Side (AAS) as different solutions to "Joe's Problem "during Lesson 2. In both cases, the students provided their solution along with supporting reasons why their proposed approach or solution was valid.

Within the teacher's consideration of students as the source of mathematical ideas, students could use mistakes as an opportunity to rethink their conceptions of mathematical ideas and as a catalyst for seeing new ideas. For example, Mr. P established the norm that all ideas were important to share but must be thoroughly examined and discussed within the group or in whole class discussion. If the proposed idea or solution contained an error, then the group members or the entire class can discuss and fix the error. For example, when a group presented Angle-Angle-Angle (AAA) as a congruency postulate. Mr. P used this suggestion as an opportunity to show a counterexample of AAA and the class discussed why it would not work. Mr. P held the opinion that shared ideas including wrong answers or procedures have the potential to generate whole class discussions. Mr. P held the expectation that students can provide answers that could be used to show inconsistencies as in the case of the AAA congruency postulate.

Responsibility for Learning

Ms. M and Mr. P both held the opinion that students were responsible for each other's learning. Students must reach a consensus when using mathematical reasoning and proof during group work. Ms. M established the norm that students must give each other feedback and collaborate their final solution when doing group work. Mr. P established the norm that the entire group must always agree on their solution before presenting to the class. Mr. P explained during one of the post-observation interview that, when he started using group work at the beginning of the school year he established the norm that governed collaborative learning. By this Mr. P and his students agreed, "it is okay for another classmate to build on or enhance another student's ideas. I really want the students to feel free to give feedback could help improve an idea or initiate another useful idea." The latter quote suggest that Mr. P gave students the autonomy to guide each other's mathematical thinking. Therefore students were accountable for the development of each other's mathematical ideas.

Ms. V also held students responsible for learning. Students were responsible for evaluating each other's ideas. Ms. V established the norm of opening up the classroom discussions after revoicing student's claims. After repeating or rephrasing a student's claim Ms. V usually would ask the other students "What are your thoughts about this?" or "Do you agree with [student's name] statement?" This question suggested Ms. V's expectation that students should provide feedback on their colleague's statements or claims. Additionally the question suggested the expectation that students must indicate their agreement with or disagreement about the proposed mathematical idea. This suggested the norm in the classroom that after the teacher revoiced a student's claims, the other students expected an invitation for them to evaluate and share their feedback.

In Ms. M's instruction she would usually redirect students questions to the wider classroom community. For example, when a student asked a question, Ms. M redirected the question by asking the other students for their opinion or feedback. Ms. M would usually repeat the student's question or claim and then ask, "What do you think?" This question extended the responsibility of providing feedback to the other students. As a result the students took on the role of the *more knowledgeable other* (Vygotsky, 1978), during the social interactions surrounding the reasoning and proof activities.

Teachers' use of Curriculum Materials

Table 3.12 shows the teachers use of the textbooks. All three teachers used the recommended textbook as a reference resource for planning their lessons or obtaining suitable examples. Ms. M stated that she used the book to teach content in particular she used the theorems and proofs in the textbook during her instruction. Ms. M also relied on the suggested investigative classroom activities for investigating theorems and concepts. Ms. M used one of these activities in Toolsie (2009) as a group activity about making a conjecture about the sum of interior angles of a triangle. Ms. M was the only teacher who expressed in her pre-observation interview that she utilized the textbook extensively to guide her lesson plan.

Table 3.12

Teachers' Use of Textbooks

Pseudonym	Teaching content	Reference Resource	Assigned Homework	Assigned in-class work
Ms. V		✓		
Ms. M	✓	✓	✓	✓
Mr. P		✓	✓	

Note: ✓ = Teacher stated that he or she made frequent use of textbook for this purpose

Both Ms. M and Mr. P assigned homework from the recommended textbooks. Both teachers assigned the practice proof questions on congruence. Ms. M also assigned one of these proof questions as in-class group work during the first lesson. Ms. V did not use the textbook for assigning homework, in-class work, or teaching content. Ms. V stated that she used external resources such various on-line recourses such as National Council of Teachers of Mathematics (NCTM) illuminations website and other textbooks to plan her lesson. Ms. V gave her students worksheets comprising of CSEC past examination questions for homework and in class work.

Table 3.13

Teachers' Use of CSEC Examination Preparatory Materials

Pseudonym	Teaching content	Reference Resource	Assigned Homework	Assigned in-class work
Ms. V		✓	✓	✓
Ms. M		✓		
Mr. P		✓		✓

Note: ✓ = Teacher stated that he or she made frequent use of CSEC exam materials for this purpose

Table 3.13 shows teachers' use of the CSEC preparatory materials. These included past examination papers, examination syllabus, and examiners reports on students' performance. Ms. V and Mr. M used the past examination questions to assess students understanding of the congruence topic during their third lesson. Both teachers also used the examination question to demonstrate how to construct an acceptable proof. In both instances the teachers outlined the requirements for an answer that would attain the best mark from the examiners. Teachers used the past paper questions to establish thee sociomathematical norm of what counts as an acceptable proof.

Discussion

My overarching question in this study asked*: What is the nature of the teaching of reasoning and proof in secondary schools in Trinidad and Tobago.* To answer this question, I observed the instruction of three teachers at three different schools. I was interested in the type of classroom practices and sociomathematical norms teachers established, teachers' pedagogical decisions, and their use of curriculum materials to create opportunities for reasoning and proof. Based on my analysis, three main ideas are characteristic of the teachers' teaching of proof. First, teachers used collaborative learning to establish characteristics of a *mathematics-talk learning community,* which include, students and teachers act as questioners during discussions, students must always explain their ideas, students are sources of mathematical ideas, and students are responsible for each other's learning (Hufferd-Ackles et al., 2004). Second, teachers provided authentic mathematics experiences through opportunities that afforded students' engagement with all the processes of reasoning and proof. Third, teachers used the CSEC examination materials as a support mechanism for creating opportunities for students to construct proof. Figure 3.37 summarizes these three ideas that were evident in all three teachers instruction during my observations.

Teachers created a social constructivist environment through group work and whole class discussions, which presented opportunities for collaborative learning. My analysis suggested that teachers used group work to help achieve the objectives of their various lessons on congruency. In each classroom, the teacher allowed students to collaborate during group work to develop a final solution representative of each group for whole class discussions. In mathematics education, research on student learning traditionally focused on the individual learner capabilities and neglected the social and cultural influences on cognition and learning (Bruner, 1996).

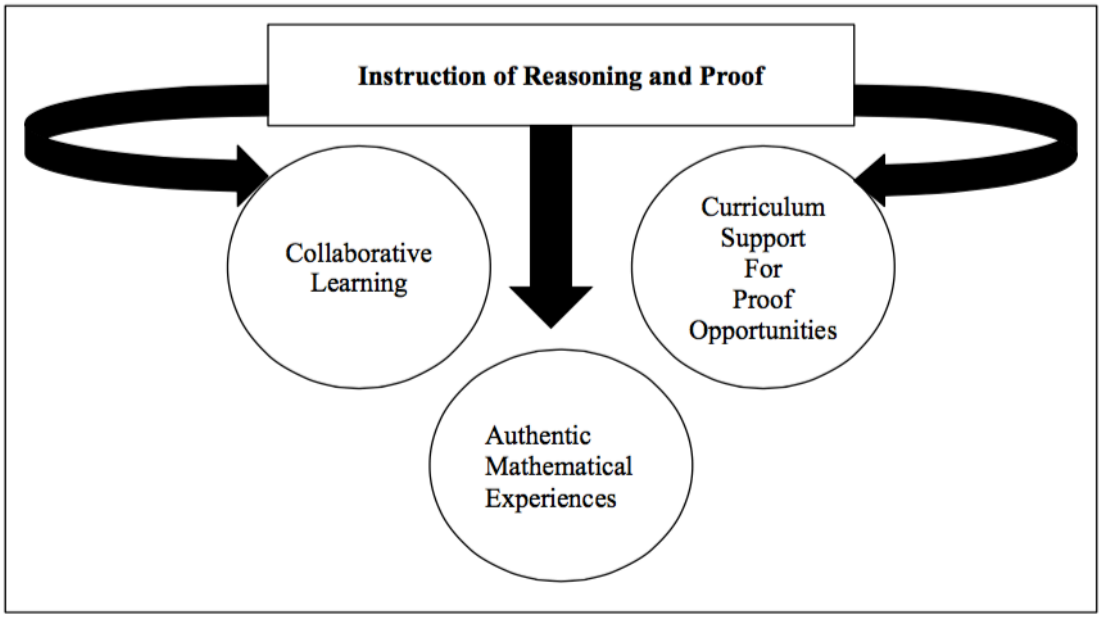


Figure 3.37: The nature of instruction of reasoning and proof in Trinidad and Tobago

Opportunities for Collaborative Learning

However, several educators are now considering the students' capabilities when they are embedded within formative and potentially supportive social and cultural settings (Conway, 2002). The focus on students' individual capabilities in mathematics education has resulted in (a) an underutilization of the student's existing knowledge, (b) unnecessary lowering of expectations about what learning is feasible in the classroom, (c) underestimation of the role of peers in contributing to learning and (d) a reliance on teaching strategies which overly compartmentalize teaching and learning activities (Greeno & Goldman, 2013). As a result, the focus in research shifted to incorporate the cognitive and socio-cultural aspects of students' learning. These aspects refer to the perspectives that students can actively gain mathematical knowledge though a process of self-organization of individual knowledge or by negotiating with knowledgeable others during social interactions within guided participation (Vygotsky, 1978).

Several researchers claim that the purpose of collaboration in the learning of mathematics is based on the notion that learning is a social process that is determined through interaction with others (Lave & Wenger, 1991; Vygotsky, 1978). Within a sociocultural perspective, learning is defined as the ways of increasing ones' participation during social activities such as group or whole class discussions (Lave & Wenger, 1991). Furthermore in mathematics education, this refers to the perspective that students' mathematical learning is defined by their increased participation in mathematical practices such as question, reasoning, and articulating one' ideas and mathematical thinking (Greeno & MMAP, 1997). Therefore from this perspective, I define collaboration as a means by which students engage in learning activities while increasing their participation in mathematical practices. Within such social activities, students encounter and resolve their misunderstanding of mathematical concepts through conversations within a community of learners (Yackel, Cobb, & Wood, 1991).

Based on my analysis of the teaching of reasoning and proof in Trinidad and Tobago, I argue that my findings suggest evidence that the teachers' instruction promotes students acquisition of knowledge through collaboration with others during group work and whole class discussions. During collaborations, mathematical knowledge is created and taken-as-shared among members of a group (Cobb, Yackel, & Wood, 1992; Simon, 1995). By taken-as-shared, I mean that the resulting mathematical ideas from students' collaboration during group work is agreed upon by all participating learners. Therefore through collaboration students participate in formulating problems, communicating their ideas and reasoning about mathematical concepts. All the aforementioned activities define students' mathematical learning. For example, Ms. V and Mr. P allowed students to collaborate with each other in their groups to determine the conditions for congruence of triangles. Ms. M used group work to help students collaborate and

construct a proof for the congruence of a given pair of triangles. In each teacher's instruction there were characteristics that helped students collaborate with each other as the communicated their ideas in a social setting. As a result of these activities groups of students responses to given tasks included ideas representative and taken-as-shared among their various group members.

Teachers' Expectations of Students during Collaborative Learning

Each teacher held the expectation that during group activities, all members of the group would work on the same aspect of the task at the same time. For example, in Ms. M's class when she inquired about a group's decision to change the orientation of a pair of triangles to determine congruence, the group members explained that they all worked on this new idea together although it seemed that one student initially proposed the idea. This action of the students suggested that the students decided to explore the suggestion about orientation at the same time. During Ms. M's instruction she established the norms that stipulate her expectations of students during group work. In particular the norm that students should work on the same portion of a group activity at the same time. Therefore the students' action described above seems to conform the norms in Ms. M's classroom.

Another of the teachers' expectation was that all group members needed to come to an agreement before proceeding to the next phase of a group activity. For example, in Mr. P's class, he continually reminded the students during the group activities that when reporting their final solution they must ensure that all group members agreed on their final response. This was evident during whole class discussions, when a student who reported on behalf of their group would briefly consult with his or her group members if they experienced difficulty in articulating the groups' response. Additionally the student consulted with group members before responding to questions about their proposed solution. This is important because it demonstrates the students

taking the responsibility of ensuring that their solutions were taken-as-shared among their group members.

A third expectation of the teachers' of collaborative learning included the shift of responsibility for learning from the teacher to the students. During each teacher's instruction, the role as facilitator and guide seemed to shift from the teacher to various students in the groups. The role of guide was evident as the more knowledgably student or expert would help monitor their group members' articulation and development of ideas. Ms. V and Mr. P explained this as their expectation of group work. Ms. V held the expectation that the opportunities for collaborative learning would help students who are less vocal and lower ability develop their ideas within a smaller community of learners rather than the entire class. Ms. V felt that the more knowledgeable student would assist these students develop their ideas and externalize their thoughts. In a similar manner to Mr. V and Mr. P established the sociomathematical norm that a valid group solution should have contributions from each student. All three teachers also established norms that governed student interaction in groups. These norms included the expectation that students must provide feedback of each other's ideas during group discussions.

The aforementioned three characteristics of the group work during the teachers' instruction are in agreement with Cobb et al. 's (1991) findings in their study about students' collaboration during group work in a second grade mathematics classroom. In particular Cobb and colleagues found that the teachers in their study expected that (a) all members of the group work on the same aspect of the problem at the same time, (b) members externalize their thoughts, including possible wrong procedures and answers, (c) members come to agreement among themselves before proceeding and (d) as instruction moved forward, the responsibility of monitoring of work was transferred gradually from the teacher to the students. In this study,

although the classrooms were at the secondary school level and in a different educational system, the teachers held similar expectations as they used group work to promote collaborative learning.

Recent reform efforts in mathematics education in Trinidad and Tobago are encouraging students to assume a more active role in their learning, to explain their ideas to one another, to discuss disagreements, and to cooperate in the solution of complex problems, while teachers participate in the design of these contexts and facilitation of this kind of activity (Republic of Trinidad and Tobago, Ministry of Education, 2009). All these ideas have implications for mathematics teaching. These include the possible shift in the classroom environment from teacher-led to student-directed; the nature of discourse and type of activities the teacher implements; the teacher and students' expectations, and the role of the teacher as facilitator and guide. My findings suggest that overall, students took responsibility for learning and actively collaborated with each other to engage in the processes of reasoning and proof under the guidance of their respective teacher. As a result, collaborative learning seemed evident in the three teachers instruction during my observations, thus aligning with the reformer's vision.

Issues with Collaborative Learning

Collaborative learning fosters " the joint production of ideas" whereby students through social interaction, listen and provide feedback to each other's idea, thus leading to a shared understanding of mathematics (Staples, 2007, p. 162). However, collaborative learning is not always an expected result of group work. It is possible that students can participate during group activities and not contribute in ways that define learning from a sociocultural perspective. For example, students may share ideas and have them dismissed by the more knowledgeable other or more dominant group members. In this case, mathematical ideas are not taken-as-shared among

group members. Nonetheless, group work creates opportunities for students to engage in collaborative learning.

However, some may disagree that group work is a pedagogical panacea (O'Connor, 1998; Noddings, 1990) thus leading indirectly to some criticisms of applying a social constructivist perspective to teaching and learning. Some students may participate eagerly while others sit and wait for others to develop the answers. Students can be rude and cruel to one another (O'Connor, 1998), and teachers have to watch group operations to be sure that their students are learning to help and care for each other rather than focus on completing assigned tasks expeditiously.

For example, social relationships can work against group sense making and the negotiation of meaning. By negotiation of meaning I mean the mechanisms of reaching a shared understanding of conceptual ideas among a community of learners. O'Connor (1998) examined this issue in the research she conducted as a participant observer in a sixth grade mathematics class over two years. Based on her observations, student interactions revealed ways in which students ideas were often discounted or dismissed due to students resistance to collaboration or their lack of respect for less dominant group members. The data in my study is consistent I with Noddings' (1990) findings, that there exist increased complexity for the teacher who must attend to socializing students about new ways of working with peers, as well as new ways of thinking about mathematical learning. Nonetheless, there is still a need to consider such interactive activities promoted by a social constructivist perspective because it allows students to see mathematics as created by a community of people (Franke & Kazemi, 2001). Additionally this perspective supports students learning by involving them in the creation and validation of ideas and it helps students become aware of more conceptually advanced forms of mathematical

activity. Most importantly it helps teachers create an atmosphere reflective of the community of practice of real mathematicians. Additionally the teachers' pedagogical approaches help them meet reform demands that support this community level of learning. The teachers' creation of opportunities reflective of the mathematical practices within the work of real mathematicians aligns with my second conclusion about the instruction of reasoning and proof in Trinidad and Tobago. This refers to teachers offering students authentic mathematics experiences.

Offering Authentic Mathematics Experiences

My analysis of the observational data suggested that the teachers offered opportunities for students to engage in authentic mathematics experiences By this I mean that through their instruction of reasoning and proof, students were able to identify patterns, make conjectures, and develop non-proof and proof arguments. My use of the word authentic is in agreement to the descriptions provided by Cirillo and Herbst (2012). These researchers considered authentic mathematics experiences as those, which allow students to make reasoned conjectures, validate claims and mathematical arguments, and construct proofs. Cirillo and Herbst also suggested that within such experiences teachers should promote and allow inductive and deductive reasoning. This is important to help students move through the processes of reasoning and proof and engagement in the work of real mathematicians.

Throughout the observed lessons the three teachers offered opportunities for students to create definitions of congruency. For example, Ms. M and Mr. P both allowed students to formulate their definitions based on empirical investigative activities. These teachers provided opportunities through recognition of characteristics of congruency shapes, or classification of shapes based on size and shape to develop hypotheses about congruence. Through the teacher's guidance, students created a generalized definition of congruent shapes. Eventually the teachers

allowed students to participate in activities that led to conjecturing about the conditions for congruence in triangles.

All three teachers used activities that helped students develop reasoned arguments to test their conjectures. For example in the problem "Slow Joe," Ms. V allowed her students to find necessary conditions for congruent triangles. Ms. V expected students to provide arguments in support of their hypotheses or conjectures. Ms. V culminated the activity by informing students that their reasoned conjectures, Side-Side Side (SSS), Side-Angle-Side (SAS), Angle-Side-Angle (ASA) and Angle-Angle-Side (AAS) were the well-known postulates of congruence. Similarly in Mr. P's class, he offered opportunities for students to develop and establish the theorems of congruence.

In Ms. M's first lesson, she offered an opportunity for students to investigate the sum of interior angle theorem. Ms. M gave her students the opportunity to develop and establish this result rather than directly telling them. Ms. M guided students through pattern identification, conjecturing, and evaluating proposed arguments for validating conjectures. Although Ms. M did not do the formal proof, similar to the other teachers, she offered the type of reasoning opportunities that allowed students to make and test their conjectures. Therefore students engaged in authentic activities of real mathematicians.

Overall, the three teachers seemed to structure their lessons to foster the type of scaffolding of students' development from inductive to deductive reasoning. Although the latter form of reasoning was not related to proving the congruence theorems, teachers provided opportunities for students to write deductive arguments using these established results to prove congruence of triangles. The three teachers used CSEC past examination questions to create the opportunities for developing deductive arguments.

Cirillo (2009) argued that if students are to engage in authentic mathematical experiences, they must have opportunities to conjecture and not just prove. Cirillo in agreement with NCTM (2000) claimed that doing mathematics as discovering, and conjecturing or "informed guessing" as a major pathway towards discovering mathematical ideas (p. 254). I concur with these arguments and I posit that the three teachers' instruction provide examples of conjecturing as a means of engaging students in the development of mathematical ideas. These examples join other cases studies such as in Cox (2004). In this study Cox, an experienced Geometry teacher, described how she motivated her students to see the intellectual need of proof in a mathematician's work. Cox explained that she achieved this objective by having her students make conjectures based on observations, test their conjectures, and then construct proof to validate the conjectures. As a result, students engaged in activities of reasoning and proof. Cox claimed that the focus on conjecturing provided the necessary scaffolding needed for students' development of proof. Additionally this focus helped students have ownership of their knowledge and appreciate the need for proof in their validation of claims, which is reflective of the actual work of mathematicians. In a similar manner, it seemed that the teachers in this study emphasized conjecturing as a means of giving students ownership of their mathematical ideas. This objective aligns with the characteristic of reform-based teaching whereby students are the source of mathematical ideas and also have the responsibility of learning (Hufferd-Ackels et al., 2004) Moreover I argue that the emphasis on conjecturing supports the creation of authentic mathematics experiences reflective of the work of real mathematicians.

Within the work of real mathematicians, mathematical knowledge develops through a process of pattern identification, conjecturing, developing non-proof arguments, and constructing proofs (Hanna, 2000; Herbst & Brach, 2006; Lakatos, 1976; Pólya, 1954; Stylianides, 2009).

Students' engagement in these activities allows them to develop mathematical arguments in similar ways as mathematicians. As a result of these authentic experiences, students develop an appreciation for mathematics and see the intellectual value of reasoning and proof in their school mathematics experiences (Otten et al., 2014). The latter is in agreement with my argument throughout the dissertation study that there exists a need for students to engage in authentic mathematical experiences and see the intellectual necessity of reasoning and proof in school mathematics. These expectations are important because they adhere to the reformers recommendations for reasoning and proof to have a pivotal role in students' mathematical experiences (Republic of Trinidad and Tobago, Ministry of Education, 2003, 2009). More importantly, the authentic experiences allow students to be prepared to participate in meaningful experiences with mathematical proofs and deductive reasoning in higher-level mathematics and acquire a metacognitive understanding of how mathematical ideas are developed and established.

Mathematics teachers and the curriculum materials they use are representatives of the discipline and serves as guide in helping students engage in activities of reasoning and proof (Bieda, 2010, Stylianides, 2009). The role of the teachers is crucial for establishing norms of proving that are characteristics of the discipline and reflective of the work of real mathematicians (Stylianides, 2009). Moreover, teachers have the task of helping students discover and learn the usefulness and intellectual necessity of these norms and authentic experiences (Herbst, 2002). Several mathematics education researchers in various countries, where the reformers' envision an integral role of reasoning and proof throughout the curriculum, such as US, Italy, France, Japan, and UK, have found that teachers face challenges when they include activities related to reasoning and proving as a formative part of their instruction (e.g., Bieda, 2010; Hoyles, 1997, Jones et, al., 2006). The challenges teachers face are primarily due to the difficulties students

encounter when they are expected to engage in the processes of generating justifications and proofs (Mariotti, 2006). Therefore it is imperative that students are continually provided with experiences with making, testing and proving their claims. These activities will help students see the need for justifying their claims and role of proof in their mathematical experiences.

According to policy documents of Trinidad and Tobago secondary school students should be given opportunities to engage in pattern identification, conjecturing, formulating proof arguments and non-proof arguments throughout their mathematical experiences (Republic of Trinidad and Tobago, Ministry of Education, 2003, 2009). Although policy makers throughout the world recommend that students have authentic opportunities to engage in reasoning and proof, Cirillo and Herbst (2012) suggest that the process of proving is more beneficial than covering a large quantity of recommended theorems and definitions. Therefore in the same manner, I concur with the suggestion that in order for all teachers to meet the reformers' demands for authentic mathematical experiences in Trinidad and Tobago, it would be beneficial for teachers to move away from simply covering mathematical content in the national curriculum, CXC syllabus, and textbooks. Teachers should include opportunities that focus on all the processes of reasoning and proof that allow students to see the process of how theorems are developed and established. These activities will be formidable in helping students see the intellectual need of reasoning and proof in their mathematical experiences.

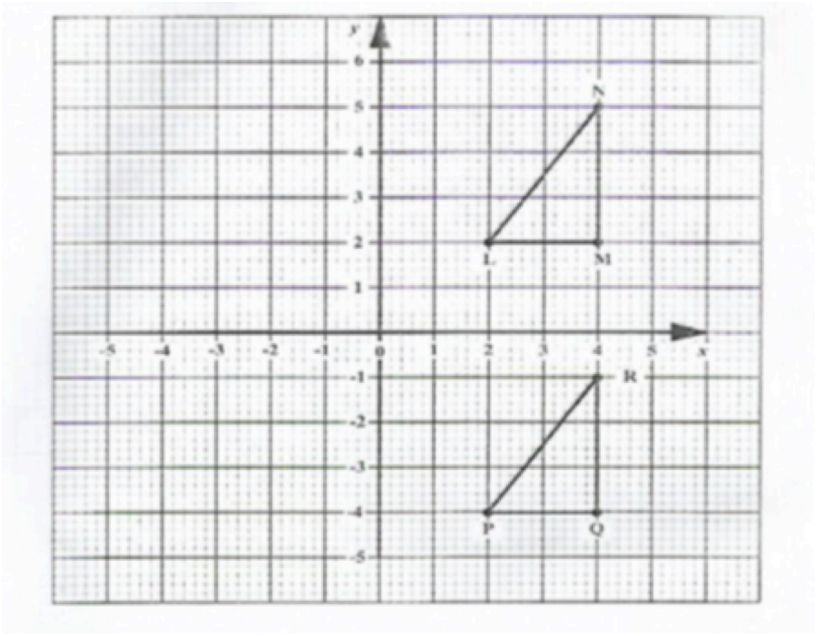
Curriculum Support for Instruction of Reasoning and Proof

During my observations, the teachers used the CSEC examinations preparatory materials to create opportunities for students to construct proof arguments. In particular, the teachers used past examination questions to create worksheets or in-class work to assess students' competencies with proving congruence. Teachers used the CSEC examination questions to help

students apply the theorems of congruence. For example, in both Mr. P and Ms. V classes, the teachers established the norms about what counts as an acceptable proof when answering CSEC examination questions. Both teachers emphasized the examiners' expectations for a valid answer of a question requesting a justification for congruence of a pair of triangles. Figure 3.37 shows a more recent exam question in the CSEC exam. As shown in Figure 3.37, the question asked students to provide reasons why the triangles resulting form the transformations are congruent. As a recent feature of the examination, students are not formally asked to "prove" but to provide reasons explaining why the congruence holds.

In the first study of this dissertation, I defined such exercises as Geometric, Calculation with Number and Explanation (GCNE) or simply calculate and explain. In these opportunities, students are required to develop arguments explaining why the given pair of triangles is congruent. Ms. V used this question in her class to help students develop rationales about congruence. Ms. V instructed her students to use their recently acquired knowledge about the conditions for congruence to construct their arguments. This aforementioned example demonstrates how the curriculum materials provided the support teachers needed to create opportunities for constructing proofs.

Mr. P similarly used the past examination questions to help students construct arguments that meet the examiners, requirements. Mr. P and Ms. V also used the examiners' report on these past examination papers for guidelines about a solution that would receive full credit. The examiners report also provided worked solution, which included a sample proof argument meeting their expectations, and would receive for full credit. Both teachers indicated that they used the examiners worked solution to help them understand the expectations.



iv) Complete the following statement:

ΔPQR is mapped onto ΔFGH by a combination of two transformations. First

ΔPQR is mapped onto ΔLMN by a ..... parallel

to the .....; then ΔLMN is mapped onto ΔFGH

by a ..... in the .....

ΔPQR and ΔFGH are congruent.
State TWO reasons why they are congruent.

.....

.....

Figure 3.37: An example of a CSEC examination past examination question. Adapted from January 2015 examination booklet.

Several researchers claim that the closer a teacher's contact with reform-oriented assessments, or learning about examiners expectations, the more likely he or she will have internal incentives to create opportunities for students to understand the requirements for success in the examinations (Cohen and Hill, 2000, Lubienski, 2011). In education systems where students' performance on external examinations holds teachers accountable to educational stakeholders, teachers use the assessments as the measurement for predicting what content

should be emphasized and taught in their classrooms. Lubienski (2011) called this "teaching to the test." Some researchers suggest that the pressure of high-stake national testing affects teaching practices differently (Lubienski, 2011; Schorr, Firestone & Monfils, 2003). For example Lubienski, (2011) found that the pressure through testing had a greater effect on the content taught than on teaching practices. Schorr et al. (2003) noted that while testing may have an effect on teaching practices, the resulting changes in instructional strategies are often not substantial to induce a long lasting effect on teaching. As a result the pressure of testing led teachers to implement more practice with examination questions rather than focusing on developing students' conceptual understanding. In some cases teachers are unable to meet the demands of policy makers, demands of schooling through testing and accountability measures, and the demands of the school environment (Kennedy, 2005). Therefore teachers make decisions that help them address the demands that are high-stake and have the greatest influence on their accountability. My analysis of the observed lessons did not provide sufficient evidence to make similar claims. However the observed lessons demonstrated that the examination materials provide support to teachers' creation and implementation of opportunities for students engage in reasoning and proof. In a future study it would be worthwhile focusing on the interplay between reformers' recommendation, testing, and instructional practices. For example future inquiry could look at how the CSEC examinations influence teacher's ability to meet the demands of the reform. Additionally the ways in which professional development supports teachers overall teaching performance and students performance on external examinations.

Insufficient Support of Curriculum Materials and Missed Teaching Opportunities

Although the CSEC materials supported teachers' creation of opportunities for reasoning and proof, there were cases when these materials did not provide the support needed by teachers.

In particular, I refer to the lack of support to help teachers explain why the Angle-Side-Side (ASS) or Side-Side-Angle (SSA) is not a condition for congruence. In a teaching episode during the lesson 2 of Ms. V's teaching of congruence, there was a missed opportunity for Ms. V to demonstrate why SSA is not a theorem for congruence. Although the proposed notation was aligned to the order of stating the measurements the students used for finding congruence, Ms. V had an opportunity to prove to students that SSA or ASS does not hold for all cases of congruent triangles (see proof in Appendix D). It is possible that during this episode Ms. V seemed to consider the issue of the order of naming the corresponding parts as crucial to students' learning of the SAS postulate. This demonstrates the case where teachers' makes decisions to go along with student-led discussions. It is possible that if Ms. V choose this opportunity to discuss why SSA did not hold, she may have (a) disrupted students' mathematical thinking, (b) subverted students' authority as the source of mathematical ideas, or (c) not adhered to her stated goal of allowing students using mistakes as an opportunity to rethink their conception of SAS theorem for congruence. However, my point here is that, through my analysis of the three recommended textbooks in Study 1 of the dissertation, neither of them provide support for explaining why ASS or SSA does not work. As a result teachers can only offer explanations based on external knowledge about these results or as in the case of Ms. V simply state it is not valid. I hold the opinion that the provision of a reason would have demonstrated that one should always proved explanations even when disproving a result.

Although this episode demonstrates a missed opportunity, more importantly it demonstrates a case of when curriculum materials could not provide the support to teachers. Research suggests that teachers' decisions about what to implement in their classrooms and how to implement it are mediated through the curriculum materials they use (e.g., Beaton et al., 1996;

Lubienski, 2011; Remillard, 2000; Stein, Grover, & Henningsen, 1996). Researchers have found that the basic features of the curriculum such as the content, organization, and sequencing of topics impact students conceptions of proof and teachers' creation of opportunities for reasoning and proof (Bieda, 2010; Chazan, 1993; Healy & Hoyles, 2000; Jones et al., 2006; Otten et al., 2014; Sears & Chávez, 2014).

Yackel and Hanna (2003) noted that the most challenging tasks of mathematics educators in their quest to help students' ability with reasoning and proof is "to design means to support teachers in developing forms of classroom mathematics practice that foster mathematics reasoning and that can be carried out successfully on a large scale" (p. 234). This quote suggests that in the case of Trinidad and Tobago, it seems that mathematics educators may have the same challenges in providing substantial support to help teachers create opportunities for reasoning and proof. Given that providing explanations in support of why a result may or may not be valid is a mathematical practice that fosters reasoning, it is essential that teachers are equipped with curriculum materials that support this type of activity. Therefore, I argue that teachers must be equipped with textbooks that provide rich opportunities for students to engage in reasoning and proof and also provide the necessary guidance to enact these opportunities.

Conclusion

In this study I examined the nature of teaching of reasoning and proof in Trinidad and Tobago. My analysis of classroom observation of three teachers suggested that teachers' instructional practices exhibit elements of reform-based instruction. These include teachers, use of open-ended and direct questions to solicit students' ideas. Additionally as in all three classes I observed, there seemed to be the acceptable social norm that facilitated students' role as questioner. For example, each teacher expected students to ask questions to clarify proposed

ideas during whole class discussions. Each teacher established social norms that governed how and when a student can ask questions. In this case, questioning helped students articulate their ideas when responding to questions and clarifying ideas when they posed a question. Teachers also established sociomathematical norms that outlined what counts as valid proof and what counts an acceptable answer. The aforementioned norms supported the expectation that students must always provide explanations for their mathematical thinking. This is another characteristic of reform-based instruction as described by Hufferd-Ackles et al. (2004).

The analysis of observations of the instruction of reasoning and proof suggest that students' learning of reasoning and proof were framed within social interaction and collaborative learning. Through several phases of group work and whole class discussions, teachers seemed to motivate students to develop their ideas from an individual to a community level. Vygotsky (1978) posited that group interaction as one source in the development of ideas; that is, he suggested that learners gradually internalize the talk that occurs in groups. During the supportive interactions with knowledgeable others students begin to challenge themselves, ask for reasons, and in general monitor their own mental work when faced with difficulty. However, it would be beneficial in a future study to investigate the dynamics of student-to-student interaction during group work. This will help in determining the extent to which the teachers' expectations are upheld during group activity. My assertions about collaborative learning are based on observations of a small sample of only three teachers. In a future study, I could investigate whether these characteristics are homogeneous among a larger sample of teachers in Trinidad and Tobago.

In line with previous literature, teachers in Trinidad and Tobago used strategies to help create a social constructivist environment for learning Teachers used strategies such as revoicing

and questioning which aligns with previous studies that examined the teaching of reasoning and proof (e.g., Bieda, 2010; Martin, et al., 2005). As a result teachers seemed to implement instructional practices that adhere to the reformers' vision for students' classroom experiences with reasoning and proof. Although the findings of this study may be limited because of the small sample, it does provide an understanding of the implications of reform in a centralized governed educational system that offers prescriptive guidance through a national curriculum materials on the teaching of reasoning and proof. A follow up study could look at these implications using other case of algebraic thinking to look at the dynamics of reform policies and practice in Trinidad and Tobago. The implications of this current study and future work will be the provision of a greater understanding of teachers' instruction of all areas of the mathematics curriculum as they meet the demands of the reform in Trinidad and Tobago.

CHAPTER 4:

SECONDARY SCHOOL STUDENTS' CONCEPTIONS OF PROOF IN TRINIDAD AND TOBAGO

Introduction

Over the past decade there has been a growing appreciation for the importance of reasoning and proof in school mathematics (Ball et al., 2002; CXC, 2013, 2014; NCTM, 2000; Republic of Trinidad and Tobago, Ministry of Education, 2009; Schoenfeld, 1994; Yackel & Hanna, 2003). The justification for this includes reasoning and proof provides the basis for mathematical understanding and proof is necessary for developing, establishing, and communicating mathematical knowledge. Despite the increased emphasis, students still demonstrate difficulties in acquiring proficiency with proof (Balacheff, 1988; Chazan, 1993; Coe & Ruthven, 1994; CXC, 2013, 2014; Harel & Sowder, 1998; Healy & Hoyles, 2000, Sowder & Harel, 1998; Stylianides, 2009). For example, students may not understand the roles of proof in school mathematics (Healy & Hoyles, 2000) or think that only statements that are already known to be true are proved (Tinto, 1988). Students may believe that empirical evidence constitutes a valid proof (Balacheff, 1988; Chazan; 1993; Martin & Harel, 1989) or that a deductive proof does not necessarily protect a statement from counterexamples (Chazan, 1993). The aforementioned findings suggest that even in the case of Trinidad and Tobago, students could possibly have difficulties with the use of empirical arguments, understanding the roles of deductive argumentation, and the necessity of reasoning and proving in their mathematical experiences.

The reformers of mathematics education in Trinidad and Tobago suggest that students should have more opportunities with proof in their secondary school mathematical experiences

(Republic of Trinidad and Tobago, Ministry of Education, 2003, 2009). Despite the calls for proof to play an integral role in school mathematics, students have difficulties in appreciating the importance of proof in their mathematical experiences and understanding the role of proof. This is evident in students opting to not answer proof-based items in the Caribbean Secondary Examination Certificate (CSEC) exams and the low performance of those who attempt these items (CXC, 2013, 2014). In Trinidad and Tobago, researchers are working to understand students' perspectives about what constitute a proof and the roles of proof in school mathematics. Researchers are also interested in how the new reform-oriented mathematics curriculum, CSEC examination materials, and teachers' instruction influence students' notions of proof in school mathematics (Bell-Hutchinson 2004; Brown, 2005). As a result of these concerns and findings about students' low performance on items in the CSEC examination, there exists a need to investigate students' perceptions of (a) the purpose of proof in school mathematics, (b) the logical structure requirements of a valid proof, (c) the generality of a proof, and (d) their opinions of the opportunities to do proofs in their textbooks and CSEC assessment materials.

In this study, I investigated students' perspectives of the nature of proof and their ways of thinking about proof. I used semi-structured interviews to gather evidence of students' conceptions of proof. I also used the findings of students' performance on proof items in CSEC assessment reports to guide the formulation of questions in the interview protocol. The analysis of the interviews could potentially help in understanding students' perspectives of the opportunities for proving in CSEC assessments and textbooks they use. Additionally, I used students as informants of the cultural notions of proving in classrooms in Trinidad and Tobago. By cultural notions I refer to the structure and purpose of proof promoted in textbooks, assessments, and their perspectives about what are the "sociomathematical norms" (Yackel &

Cobb, 1996) their teacher establishes during the teaching of proof in Trinidad and Tobago. In the next section I discuss students' performance in the CSEC mathematics examinations and how this influences my inquiry into students' conceptions of proof.

Students Performance in CSEC

In recent years there has been the prevalence of reasoning and proof items in the CSEC mathematics examination Geometry questions (CXC, Subject Award, 2013, 2014). Figure 4.1 shows a typical exam question in the January, 2014 examination.

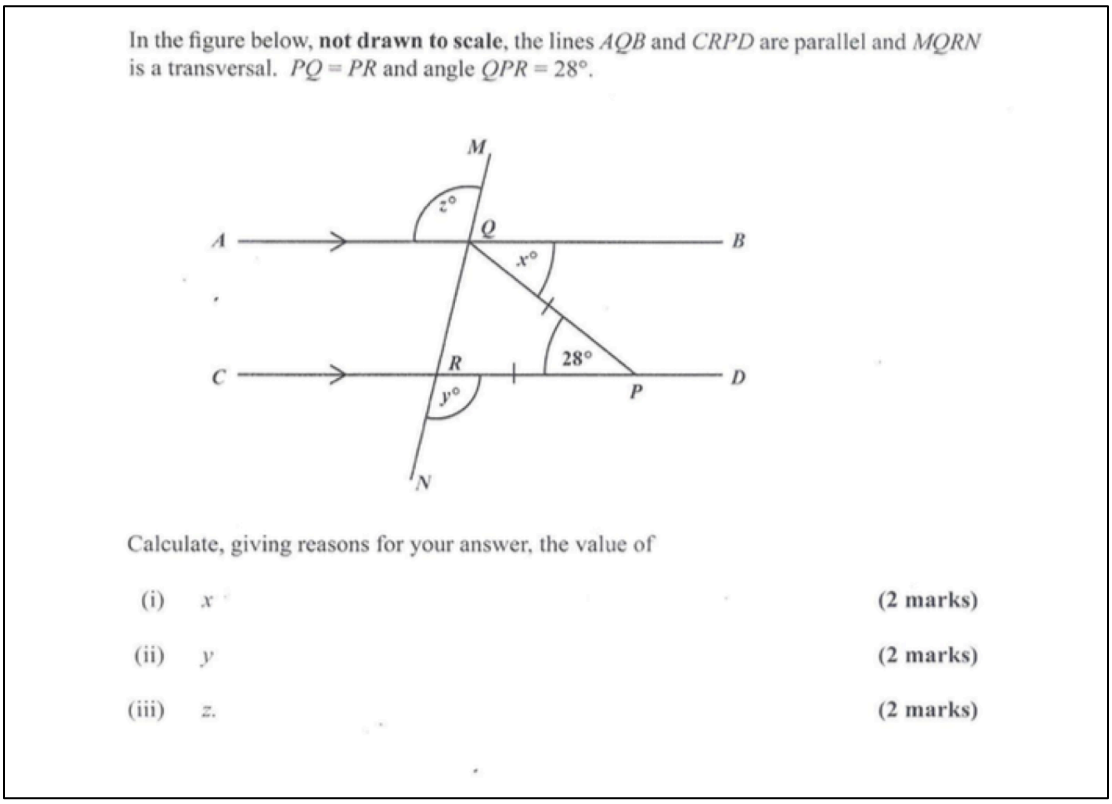


Figure 4.1: An example of reasoning and proof question from CSEC January 2014.

In this question, students are asked about finding the unknown angles *x, y,* and *z* in the given diagram. Students are expected to use the given information that the lines *AQB* and *CRPD* are parallel where *MQRN* is a transversal (cutting across the aforementioned parallel lines) to help

solve for the unknown angles. Additionally students are provided with the measure of angle *QPR* and the fact that the lines *PQ* and *PR* are of equal length. This question required that students calculate the unknown measures and provide reasons justifying their computations.

According to the examiners' report, this question assessed students' ability to determine the measures of angles using the properties of parallel lines and transversals. Of the 94% of students attempting this question, less than 1% earned the maximum marks. The examiners reported that students' performance generally was unsatisfactory. Overall, students were able to correctly state the values for the unknown angles in part (i) angle *x* and part (ii) angle *y*. However, over 90% of these students were not able to state the value of angle *z* in part (iii). In the cases where students were able to identify the measures of all the unknown angles correctly, they could not provide reasons to support their calculations. The examiners recommended, "candidates [students] should be drilled in the practice of stating reason or reasons for answers derived from the Geometry of plane figures" (CXC, 2014, p. 8,). The aforementioned quote suggests that teachers should provide opportunities for students to develop supporting arguments, which use their Geometry knowledge to justify the validity of calculations they perform. The examiners also recommended that teachers should encourage their students to use mathematical terms to describe the geometrical relationships they observe or derive from geometrical figures.

Although this question did not explicitly ask students to prove, students were expected to provide supporting arguments using the properties of lines and transversals to justify their calculation. A typical response to this question can be considered as an opportunity for reasoning and proof, I define as a Geometric Calculation with Number and Explanation (GCNE). A GCNE is a geometric calculation that requires the use of geometrical theorems and definitions to explicitly provide reasons to support the result of the computation. Therefore it is of interests to

this study that with the prevalence of such opportunities in the CSEC exams, I investigated whether students realized that these informal arguments could be considered possible opportunities to construct proofs. Furthermore according to the policy documents, students should be proficient in creating informal justification and proof arguments to explain why a mathematical result holds (Trinidad and Tobago, 1994, 2005, 2006).

The current CSEC mathematics curriculum also supports the policy makers' advocacy by stating that students should be able to construct mathematical arguments and critique the arguments of others (CXC, 2012). The issue with this requirement is that it entails some measure of providing supporting reasons to justify claims or geometric calculations. The development of informal arguments promotes the explanatory nature of proof in school mathematics. This is important in the context of proof and reasoning because it provides the scaffolding needed to motivate students to develop their informal arguments into logical deductive arguments supporting their mathematical claims. By providing these reasons, students can develop metacognitive skills in making connections to Geometry content when solving for an unknown quantity in a given geometric object. As a result, I used this question in Figure 4.1 in my interview protocol to determine students' opinions about whether a response to a GCNE question would be considered a proof.

The policy documents in Trinidad and Tobago, recommend that that students should be given opportunities engage in critical thinking and reasoning. Teachers should guide students in the processes of making sense of mathematical ideas and constructing logical arguments (Trinidad and Tobago, 1994, 2005, 2006). This recommendation suggests that students' opportunities to learn proof will directly impact their achievement of these learning objectives. Students' opportunities to learn proof will be influenced by several factors. These factors could

include the emphasis textbooks and teachers place on proof, the time teachers allocate for proof-related and activities, teachers' perspectives of student learning whether constructivist or behaviorists, and students perspectives about proof. Additionally, the type of tasks teachers and textbooks offer students; will, to some extent, affect students' learning about proof and their performance on proof items in the CSEC examinations. However, researchers are yet to confirm these hypotheses regarding students' learning of proof in Trinidad and Tobago. Thus, the findings of this study will help in understanding the role of assessments, the content in textbooks and examination materials, and the role of teachers as agents of the reform policies in framing students' conceptions of proof in Trinidad and Tobago.

Research Questions

The research questions, which drive my inquiry, are:

RQ.1 How do students in Trinidad and Tobago view the purposes of proof in secondary school mathematics?

RQ.2 How do students view the generality and validity of a proof?

RQ.3 What are students' conceptions about the type opportunities for proving in Geometry in the CSEC examinations?

The first two research questions are important in understanding the specific ideas students in Trinidad and Tobago have about what constitutes a proof and its role in mathematics and school mathematics. The first research question addresses the students' perspectives about the purpose of doing proof in secondary school. This question helps us understand if there is any agreement between students' views of the necessity of proof and the reformers' vision that proof should have an integral part of students' mathematical experiences. This question also helps in understanding the uses of proof for students' mathematical understanding. For example, past

research demonstrates that students may see proof for its role in verifying the truth of a well-established theorem as opposed to its explanatory value in explaining why a theorem is true (Hanna, 2000; Healy & Hoyles, 1998). Students may possibly see the role of proof in discovering mathematical knowledge. This latter role agrees with the reformer's vision of students' active engagement in the construction of their own knowledge within a constructivist approach to learning.

The second research question specifically helps in gathering evidence of students' learning of proof in terms of their ideas about the role of empirical and deductive arguments in demonstrating the validity and generality of a proof. The first two research questions coupled with the third research question will help us understand specific cultural practices associated with constructing proofs in Trinidad and Tobago. Additionally we can gain insight into students' perspectives of the type of sociomathematical norms their teachers establish during proving activities. The final research question helps determine the possible role of the terminal assessments in students understanding of proof. In particular, the final research question helps us uncover possible explanations for students' low performance on proof items in the CSEC. This can be helpful for future assessments of students' ability to do proof. We can also determine the extent of the opportunities to do proof in the assessments and textbooks in supporting students' learning of proof. In the next section, I review past research about students' perspectives of proof that are pertinent to my study.

Literature Review

Several studies from different countries indicate that many students demonstrate deficiencies in one or more facets of proof competency (e.g., Healy& Hoyles, 2000; Kunimune, Fujita & Jones, 2009; McCrone & Martin, 2009). These studies provide evidence that even

though curriculum materials offer opportunities for students to engage in reasoning and proof, students still possess deficiencies in these areas. In particular, two studies, Healy and Hoyles (2000) and McCrone and Martin (2009) contributed significantly to the field and also guide this study. These two studies investigated students' understanding of proofs, their ability to construct proofs, and their views on the roles of proof in school mathematics.

In their empirical study, Healy and Hoyles administered surveys to 2459 high achieving 14- and 15-year old mathematics students in various schools across England and Wales. The results suggested that even these high achieving students had difficulty in generating proofs. The students did not demonstrate proficiency in constructing mathematical proof in algebra and were more inclined to rely on empirical arguments. Nonetheless, most of the students were aware that a statement is valid once it is proven for all cases.

McCrone and Martin (2009) found similar results from interviews with eight students from the second year of a 3-year study. Their findings were supported by data from classroom observations and questionnaire responses from over 100 students in proof-based Geometry classes in the United States. The eight students differed from those selected in Healy and Hoyles' (2000) study in that they represented a wide range of ability levels as determined by their teachers and the researchers.

In both studies, students were provided with a written proof questionnaire so as to determine their perspectives about the generality, validity, and roles of proof in school mathematics. A noteworthy feature of these studies was the inclusion of items asking students to select a correct proof from a list of argument examples. The examples in Healy and Hoyles reflected the four types of proof levels of students' proof schemes in Balacheff (1988). These levels include the different type of arguments students produce when constructing a proof. The

arguments range from the lowest level "naïve empiricism " where student use examples to prove a result to the highest level "thought experiment" where they use a deductive formal proof that holds for all classes of geometrical objects or situations. The second and third levels reflect the use of representative examples (a crucial or generic example) that seem to hold for all cases or possess common properties among the majority of cases. A crucial example (second level) is a carefully selected example that allows a student to notice important generalizable geometrical relationships. A generic example (third level) represents a class of objects and can be used by a student to make deductions about generalizable characteristics of all objects represented by this example. Healy and Hoyles provided sample arguments representing these four levels as choices for the students' selections as valid proofs. In McCrone and Martin's (2009) study, they used sample arguments that reflected the two types of reasoning students use to construct an argument (inductive vs. deductive).

In these two studies, students were able to identify a correct proof, however non-disciplinary perspectives influenced their selections. For example, many students took into account, which arguments their teacher will most likely consider correct or would receive the highest grade from their teacher. Students considered that their teacher would more likely accept arguments presented in a formal style, this being either narrative paragraph or T (2-column) – format (Herbst, 2002)[14]. However, when asked for their own preferences, students agreed on arguments that were more explanatory in nature. These findings in both studies suggest that students do understand somewhat the explanatory role of proof in school mathematics. However,

[14] In the T or 2-column format, one draws a long horizontal line and a vertical line downward from the middle to form a letter T, creating two columns under the horizontal line. In the left column, one writes a deductive sequence of statements leading to the statement to prove, numbering each statement. For each step of the deduction one has to write in the right column a reason for the deduction with a corresponding number.

students did identify that empirical arguments were more reflective of what they would reproduce for a proof. It is of interest to my study that I similarly investigate the criteria students in Trinidad and Tobago use to identify a valid argument. As a result, I consider the selection of pre-written arguments aligned with the approaches in the aforementioned studies to evaluate students' conceptions of validity and generality of proof in Trinidad and Tobago.

In considering students' perspectives on Geometry proofs, past studies demonstrate that students either do not understand the role of proof in their mathematical experiences or do not appreciate the necessity of proof in school mathematics (e.g., Chazan, 1993; Herbst & Brach, 2006; Kunimune, Fujita, & Jones, 2009; Schoenfeld, 1998). For example, Kunimune, Fujita and Jones (2009) reported that although most 14 –15 year-old students in Japan (in the third year of secondary school) could write a proof, around 70% could not understand why proofs were needed. The students felt that the purpose of writing proofs was for demonstrating one's knowledge of previously taught theorems. This finding is in agreement with other studies, which also found that Geometry students view opportunities for reasoning and proof as arbitrary exercises (e.g., Chazan, 1993; Fischbein, 1982; Herbst & Brach, 2006; Schoenfeld, 1998, Tinto, 1988). In Herbst and Brach's (2006) study with 16 high school students enrolled in two accelerated Geometry classes, the students claimed that the purpose of reasoning and proving was to provide opportunities for them to showcase their reasoning and communication skills. In Schoenfeld' s (1998) study, the students also viewed Geometry proofs as simply arbitrary exercises in logic that merely confirm results already known to be true. Additionally in Tinto (1988) students viewed reasoning and proving activities as exercises imposed on them by their teacher or textbook. Finally McCrone and Martin (2009) found that students in their study also considered the purpose of proof to be applying recently learned theorems and not as a

mathematical process for establishing the truth of theorems. Although the aforementioned studies were conducted with different populations of students from different educational systems and background, these findings provide evidence that students may not see the intellectual need for deductive arguments in their mathematical experiences.

Other studies also indicate that students may not also fully understand the role of deductive arguments in proving (e.g., Chazan, 1993; Fischbein, 1982; Healy & Hoyles, 2000). These studies show that students still considered the possibility of finding a counterexample after seeing a correct deductive argument. This indicates that students fail to see the intellectual link between deduction and justifying general claims. Students may not understand the pivotal role a valid deductive argument plays in assuring the claim holds for all cases.

The aforementioned findings indicate that students may not perceive the intellectual role of reasoning and proof in mathematics. These findings highlight the potential danger of students not appreciating the necessity of reasoning and proof in their mathematical experiences. Research has shown that such perspectives can be unfavorable to students' future learning of mathematics (e.g., Muis, 2004). Therefore it is important that in the case of Trinidad and Tobago, that we investigate Geometry students' views of reasoning and proof in school mathematics. For example, students may consider reasoning and proof as an arbitrary activity imposed on them by their teacher, examiners, and textbooks. This could be a possible explanation for their low performance on proof-based items in CSEC examinations. Furthermore, the findings of my study can potentially help us determine whether students in Trinidad and Tobago see the intellectual need for reasoning and proof in their mathematical experiences.

Theoretical Framework

In this study I use a framework previously used by McCrone and Martin (2009) and Dreyfus and Hadas (1987) to help investigate students' conceptions of proof. This framework (see Appendix E) entitled, *The Six Principles of Proof and Understanding* outlines the following principles students should understand of proof: (a) the purpose and function of counterexamples, (b) roles of proof, (c) generality of proof, (d) validity and logical structure of proof, (e) logical equivalence, and (f) the use of diagrams during proving. These principles were first introduced by Dreyfus and Hadas (1987) and then further revised by McCrone and Martin (2009). The principals were initially described as common concepts and statements related to proof.

According to Dreyfus and Hadas these principles are "not well understood by most students of average ability" (p. 48). As a result, these principles are useful in guiding my analysis of students understanding of proof in school mathematics. The list of principles in this framework represents statements about proof, which would be commonly shared by any informed member of a mathematical community such as a classroom (McCrone & Martin, 2009). In this study, I use these principles to guide the design of my interview protocol for students. For example, my anticipated interview questions will obtain evidence of students understanding of proof according to the principles looking at the validity, purpose, and generality of proof. In a subsequent section I provide more detail of the interview protocol. Overall, I find this framework useful because it covers the areas of my inquiry about students' perspectives that are beneficial to understanding the larger issues of reform and its implications for students' learning of proof.

Methodology

In this study, I conducted semi-structured interviews with 21 students (male or female who are 13 to 16 years old) in forms three (ages 13 to14), four (Ages 14 to 15), and five (ages 15 to16) from three selected school sites. Because each of the school sites for this study corresponded to the schools I selected for another observational study, I selected students from the same classes I observed at the respective school sites.

Participants

For each school, I randomly selected seven students from those who provided voluntary assent and parental consent to participate in the student interviews. Additionally these students also participated in the proof-based lesson I observed in the second study of my dissertation. The selection of the seven students from each school was according to the grade level I observed at the respective school. Table 4.1 shows the demographics of the 21 students who participated in this study.

Table 4.1 lists the students' pseudonyms, respective school, class or grade level, and anticipated college major. The latter characteristic was important so as to ascertain students' future plans for pursuing mathematics at higher levels such as Form 6 (Ages 16-18 years old) in preparation for the Caribbean Advanced Proficiency Examination (CAPE) or college. This characteristic is important because it provides an understanding of how student's current perspectives about proof may impact their future mathematical experiences with proof. Additionally, by understanding students future mathematical plans, one can understand the trajectory of their learning with proof and the impact of their perceptions of CSEC proof opportunities on their preparation for their future career plans.

Table 4.1

Demographic of Students Participating in Study

School	Pseudonym	Class/Grade Level	Anticipated College Major or Career Plans
A	Cece	Five	Social Work
A	Faye	Five	Engineering
A	Jordan	Five	Engineering
A	Kelly	Five	Psychology
A	Mary	Five	Law
A	Melissa	Five	Linguistics
A	Tia	Five	Law
B	Danni	Four	Accounting
B	Kathy	Four	Mathematics
B	Morgan	Four	Accounting
B	Noel	Four	Accounting
B	Oliver	Four	Military
B	Raegan	Four	Finance
B	Taylor	Four	Finance
C	Ava	Three	Mathematics
C	Dana	Three	Mathematics
C	Delina	Three	Chemistry
C	Ray	Three	Engineering
C	Zafiyah	Three	Biology
C	Shane	Three	Engineering
C	Shanaz	Three	Accounting

Of the students participating in this study, their anticipated future careers plans varied from Business, Engineering, Social, and Natural Sciences. The Business careers included anticipated majors in Accounting or Finance. Of the 21 students, 5 anticipated majoring in this area. With regard to Engineering, four students expressed interests in majoring in this field at college. In the Social Sciences, five students expressed interests in majoring in Sociology, Law or Psychology. One student expressed interest in a career in Humanities, particularly studying Linguistics. In the Natural Sciences, two students wished to pursue college majors in this area. One student expressed interest in Biology and the other in Chemistry. Of the 21 students, 3 students expressed an interest in majoring in Mathematics at college. However all the students

expressing their interests in engineering or accounting acknowledged the expectation of possibly doing an extensive amount of required mathematics courses at college. In addition to the aforementioned areas, only one student expressed interest in a military career. However this student also identified the need to be successful in mathematics in order to pursue his future career plans.

Of the 21 students interviewed, 15 planned to pursue higher-level mathematics courses for CAPE in Form 6. The remaining six students did not wish to further their mathematics studies after writing CSEC examination in Form 5. Of the 21 students, there were 5 boys and 16 girls. The higher percentage of female participants was due to one of the school sites being a single sex (all female) school. Nonetheless, the 21 students provided a representative random sample of the population of secondary school students in Trinidad and Tobago.

Interviews

Each interview lasted 45 minutes to 1 hour. I conducted the interviews after the teaching of the reasoning and proof-based lessons on Congruency of Triangles, I observed for another study in the dissertation. The interview protocol (See Appendix F) was divided into four sections. Each section addressed principles within the theoretical framework that were useful to my inquiry.

In Table 4.2, I show how the interview questions aligned with the various principles in the theoretical framework. The principles within the framework are in column one. My anticipated questions for determining students' perspectives of the respective principle of proof are in column two. For example, in section one, I investigated the roles of proof in mathematics and school mathematics (Principle 2). My intention was to gather evidence of students' conceptions of possible roles of proof in the work of mathematicians and their secondary school

mathematical experiences. I asked students questions such as, "In your opinion, why do you think mathematicians write proofs?" Or "Could you provide reasons why you are taught proof in mathematics?" I also attempted to get their criteria for a correct or convincing proof by asking questions such as, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" I gathered evidence of students' opinions of validity of a proof (Principle 1) and logical structure of a proof (Principle 4).

In section two, I investigated students' perspectives about the use of counterexamples and the logical structure and validity of a proof (Principle 1 (b) and 4). I was interested in gathering evidence of students' knowledge about the differences between hypotheses and conclusion. I also wished to understand students' perspectives about the role of counterexamples in proving and how the existence of counterexamples affects the generality of a proof. Although the theoretical framework contained six principles, I was only interested in principles two, three, and four as indicated in columns one of Table 4.2.

Table 4.2

Examples of Interview Questions According to the Principles of Proof Understanding

Principle of Proof Investigated	Description of Principle	Examples of Anticipated Interview Question
Principle 2: The roles of proof in mathematics and school mathematics	One purpose of proof is to establish the validity of a statement.	Q3. In your opinion, why do you think mathematicians write proofs?
	Another purpose of proof is to provide insight into why the statement is true	Q5. In your opinion, could you provide reasons why you are taught proof in mathematics?
Principle 3: The generality of a proof	One method of proving a general statement is to show that the statement is true for every conceivable instance.	Consider the following arguments for proving the theorem in Example 1. The theorem is taken from Toolsie (2009) *Mathematics a complete*

	If a proof is not exhaustive then a direct proof, indirect proof, or visual proof must be constructed so that all statements or diagrams refer to all classes of figures that satisfy the given conditions.	*course.* Q9. Which argument would be closest to what you would write? Please explain. Q10. Which arguments do you suppose would receive the best mark from your teacher? Please explain.
Principle 4: Validity of proof and logical structure	Empirical evidence of one or more cases does not constitute a proof. Checking a few specific critical cases does not constitute a proof. The statements in a proof must be ordered to conform to the laws of reasoning to provide a path from the hypothesis to the conclusion. Inherent logic determines the validity of a proof, not ritualistic aspects of the form in which it is presented.	Q4. What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?

In section three of the protocol, I provided students with examples of arguments for proving a theorem taken from Toolsie (2009). The selected arguments varied in the structure and the nature of reasoning. The first argument was empirical in nature whereas the other three arguments were deductive. These four arguments varied in their generality and validity. Similar to Healy and Hoyles (2000), I used Balacheff's (1988) taxonomy of students' proof ability to guide my development of the four arguments I used in the interview protocol. My intention was to ascertain students' opinions about the generality and validity of arguments that would be their

personal choice and would provide the highest grade from their teacher and external examiners. I also wished to have argument options reflective of the varying levels of proof ability as in Balacheff (1988). The analysis of students' selections could help inform researchers of students, teachers, and external examiners criteria for a valid proof in Trinidad and Tobago.

In addition to my alignment of the research questions to the various principles of proof understandings, I designed the questions to address particular conceptions of proof. I was particularly interested in the following categories of students' conceptions, (a) the roles of proof in mathematics, (b) the roles of proof in school mathematics, (c) structure of a valid proof, (d) students and teachers' criteria for generality and validity, (e) CSEC opportunities for proof, and (f) classroom practices for the teaching of proof. In Table 4.3, I present a description of each conception and a listing of the interview questions that I anticipated would provide evidence of each respective student conception. The aforementioned six categories of conceptions shown in column one of Table 4.3 are important for two reasons. Firstly, evidence of these conceptions with regard to students in Trinidad and Tobago could add to the existing literature about students' conceptions of proof. Secondly, these categories could provide an understanding of students' ideas about teaching and learning of proof in secondary school as it relates to the recent reform policies in Trinidad and Tobago.

Table 4.3

Examples of Anticipated Interview Questions Aligning with Students' Conceptions of Proof

Category of Student Conception	Description of Student Conception	Anticipated Interview Question
Roles of proof in mathematics	The roles of proof in the work of mathematicians	3. In your opinion, why do you think mathematicians write proofs?
Roles of proof in school mathematics	The roles of proof in students' secondary school mathematical	5. In your opinion, could you provide reasons why you are

	experiences	taught proof in mathematics?
Structure of a valid proof	The characteristics of what students consider a valid proof	4. What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?
Students perceptions of teachers' criteria for generality and validity	Opinions of their teachers' expectations of the characteristics of a valid proof.	Consider the following arguments for proving the theorem in Example 1. The theorem is taken from *Mathematics a complete course* (Toolsie, 2009) 9. Which argument would be closest to what you would write? Please explain. 10. Which argument do you suppose would receive the best mark from your teacher? Please explain.
CSEC opportunities for proof	Opinions about the opportunities for proof in the CSEC mathematics examination.	14. Consider the question in Example 2. This question is taken from the January 2014, CSEC Mathematics examination. In your opinion why or why not would you consider an answer to this question as a proof?
Classroom practices for the teaching of proof.	Students as informants of teachers' use of proof-related materials and pedagogical practices during the teaching of proof.	11. Which textbook do you currently use for your form 4 and form 5 classes? Can you tell me more about the text and about the proofs they present to you? 12. Does the textbook offer questions where you can practice proofs? Can you show me a few examples if possible? 13. Does your teacher usually assign proofs from the textbook? If yes could you show me a few examples if possible?

16. In your opinion, do you think that you are sufficiently prepared for higher-level mathematics involving proof?

In the final section, I asked students questions about the type of opportunities for reasoning and proof in their textbooks and the CSEC examination. I asked questions such as "Does the textbook offer questions where you can practice proofs? Can you show me a few examples if possible?" I also asked students about the teaching of proof in their classroom. For example, I asked, "Does your teacher usually assign proofs from the textbook? If yes, could you show me a few examples if possible?" My goal in this section was for students to be informants of their teachers' classroom practices during the teaching of proof.

Coding and Analysis of Data

I transcribed and coded the student interviews according to the statements of the various principles of proof and understanding in the theoretical framework. I coded the interview data in three dimensions. In Figure 4.2, I present my taxonomy of coding categories. The columns represent the three dimensions of my coding and analysis. The rows represent the codes associated with each dimension.

Each dimension in Figure 4.2 corresponded to a stage in my coding and analysis of the data. Dimension one corresponded to the initial codes I labeled as the six categories of students' conceptions I considered in this study (see Table 4.3, column one).

Dimensions of Coding		
Dimension 1	Dimension 2	Dimension 3
Roles of proof in mathematics	a. Verification	i. Convincing oneself ii. Convincing others
	b. Explanation	i. Explanation as an expectation of the discipline of mathematics
	c. Systemization	
Roles of proof in school mathematics	a. Promoting Understanding	i. Insight into why a result ii. How to apply a result in problem solving
	b. Appreciation	i. Axiomatic system of mathematical results ii. Usefulness of other mathematical results
Structure of a valid proof	a. Correct ordering of steps b. Internal logic c. Level of completion d. Use axioms and previously proven theorems	
Students perceptions of teachers' criteria for validity	a. Include all steps b. Internal logic c. Use axioms and previously proven theorems d. General (i.e., holds for all cases)	
CSEC opportunities for proof	a. A calculate and explain problem is a proof b. Explanations of why is a proof	
Classroom practices for the teaching of proof.	a. Teacher uses textbook to teach proofs	i. Teacher never uses the textbook ii. Teacher always uses the textbook

		iii. Teacher uses the textbook to assign homework questions only iv. Teacher uses the textbook for individual seatwork or group work
	b. Teacher uses CSEC exam preparatory material	i. To practice proof questions ii. Teach examination strategies for answering questions

Figure 4.2: Taxonomy of coding categories for student interview data.

With regard to dimension two, I created the respective codes after my first phase of analysis of the data with the initial codes in dimension one. For example in category one, I was interested in students' conceptions of the roles of proof in mathematics. I labeled the dimension one code as the roles of proof in mathematics. Based on principle two of the theoretical framework, I selected the roles of verification and explanation as subthemes of dimension one. I used these two roles as dimension two codes in the taxonomy. After further analysis of the data, I discovered emerging themes of the role of verification. The emerging themes evident in students' talk were, (a) verification as a means of convincing oneself and (b) verification as a means of convincing others. Based on the evidence suggested from analyzing students' talk, I labeled these emerging themes as codes within dimension three.

After my initial coding and analysis of the data for the roles of proof in mathematics, I discovered systemization as an emerging subtheme in students' talk about the roles of proof in mathematics. The role of systemization refers to the organization of mathematical concepts into an axiomatic system of results. I considered this as an additional subtheme of the roles of proof

because this role was distinct to the other codes in dimension two. Therefore I added this code to dimension two for the roles of proof in mathematics category.

Overall each dimension corresponded to either initial or emerging themes from my analysis of the data. The codes in dimension two were either developed from definitions in the theoretical framework, or from several phases of my analysis of the data. The emerging themes were only evident from dimension two after the initial phase of my analysis of the data. For example, under the dimension one category of the role of proof in school mathematics, the second dimension category of promoting understanding had emerging subthemes resulting from further analysis of the data. These subthemes suggested the classification of promoting understanding into, (a) insight into why a result is true and (b) applying a result in problem solving. As a result I considered the aforementioned emerging subthemes as codes for dimension three in my taxonomy.

Analysis Methods

To answer the first research question about the purposes of proof in school mathematics, I considered the statements provided in principle two of the theoretical framework. According to principle two there exist the roles of verification and explanation in mathematics (McCrone & Martin, 2009). The role of verification establishes the validity of a mathematical claim or result. Whereas, in its explanatory role, proof provides insight into why a result is valid as opposed to verifying it is true. As mentioned earlier, I used these two roles as initial codes for students' responses to the following questions, (a) In your opinion, why do mathematicians write proof? (b) In your opinion, could you provide reasons why you are taught proof in mathematics?

I considered responses to the first question as describing the role of proof in mathematics and the second question addressing the roles of proof in school mathematics. I made this

distinction so as to ascertain whether students considered proof differently in the work of mathematicians as compared to their work in school. I also paid attention to possible similarities of roles evident in the students' responses. After establishing the dimension one codes, I used the definitions of the roles taken from the theoretical framework and coded the responses according to dimension two coding categories. In the case of the verification role of proof, further analysis of the data suggested emerging themes, which I labeled as dimension three codes. I used the codes within my taxonomy to code the data.

Example of Coding of the Roles of Proof in Mathematics

In the following excerpt, I present an example of a student's response to question three of the interview protocol. When I asked Ava, "In your opinion, why do mathematicians write proofs?" she stated:

> Ava: Well, when they [mathematicians] write proofs it is really showing others a confirmation that the claims they made are true.

Ava responded to my question inquiring about her conceptions of the roles of proof in mathematics. As a result, I coded this response with the dimension one code of the roles of proof in mathematics. To further analyze this quote I considered the definitions of the role of proof in the theoretical framework. Because Ava talked about mathematicians confirming the validity of their claims, I coded this quote as the verification role of proof according to dimension two. When Ava stated, "they [mathematicians] write proofs it is really showing others a confirmation" she suggested that the role of verification is a means of convincing others that the mathematicians claims are true. As a result I also coded this quotation as evidence of the third dimension code of convincing others.

Students' Conceptions of Generality of a Proof

To gather evidence of students' opinions about the generality of a proof, I provided an opportunity for students to select a valid proof from pre-written arguments during the interview. My objective was for students to have an additional opportunity to select an argument reflective of their own conceptions of validity and generality (i.e., their personal proof choice). I also intended for students to select an argument that reflected their teacher's criteria for a valid proof argument that would receive the highest mark (i.e., best mark proof choice). This latter intention was important because it addressed the possibility that the school environment and teachers' expectations could impact students' understandings of proof. Additionally, in a similar manner as previous studies, which examined students' conceptions, I offered students the opportunity to evaluate pre-written proofs. For example, as in Healy and Hoyles (2000) and McCrone and Martin (2009), I offered arguments representative of inductive and deductive approaches to proving a theorem. These studies used Balacheff's (1988) levels of students proof ability to guide the development of sample proof arguments for students' selection. In the same manner, Balacheff's taxonomy of students' proof ability guided my development of the sample arguments for students' selection. My objective was to gain an understanding of student's criteria for arbitrating the validity of an argument. In particular I paid attention to their respective criteria related to the generality of the argument.

To obtain evidence of students' perspective of generality and validity of a proof, I asked questions nine and ten of the interview protocol. In question nine, I asked students, which of these arguments would be typical to one they would write. My objective for asking this question was to gather students' opinions of structure, validity, and generality of a proof. In question ten, I asked students to select the argument that would be their best mark proof choice. By best mark

choice, I mean the proof that their teacher would award with the highest mark from the available options. In these two questions, I presented four arguments for proving a theorem taken from one of the textbooks, Toolsie (2009), used in Trinidad and Tobago (see Interview Protocol in Appendix F). The four arguments varied in their validity and generality. In Figure 4.3, I present the theorem and the four arguments for students' selections as their personal or best mark proof choice. I wished to determine whether there existed any similarities or differences between students' criteria and their perceptions of their teachers' criteria for the structure of a valid proof. Again based on the selection of the argument and their supporting reasons for the selection, I gathered evidence of students' conceptions of the generality and validity of a proof.

The theorem I used in this study is referred to as the exterior angle theorem in Toolsie (2009). The theorem stated, "if any side of a triangle is extended then the exterior angle so formed is equal to the sum of the two remote interior angle measures" (p. 444). The first argument I presented was empirical in nature and considered a few empirical cases of triangles. Similar to Healy and Hoyles (2000), I created an argument that represented the first level of Balacheff's (1988) taxonomy of proof, (i.e., naïve empiricism). I designed argument one in my protocol by using a few generated examples along with measurements to demonstrate that the theorem is true.

The author of the textbook, Toolsie (2009), wrote argument two. This argument represented the thought experiment level in Balacheff (1988) taxonomy of proof. A thought experiment is the fourth and highest level of Balacheff's taxonomy of proof. A proof at this level allows students to go beyond examples to categorize objects into sets, whereby they can extract generalizable conceptual relationships to construct a logical general deductive argument as a proof. Analogous to Healy and Hoyles (2000), I included a proof representative of the thought

experiment. Additionally as in McCrone and Martin (2009), this proof also represented what they considered a deductive approach to proving a theorem.

The author of argument two categorized triangles into two categories (a) acute-angled triangles and (b) obtuse-angled triangles. In a triangle belonging to the first category, all interior angles would be acute (i.e., less than 90^0). For a triangle belonging to the second category, at least one of the interior triangles would be obtuse (i.e., greater than 90^0 and less than 180^0). However, a right triangle would not fall in the two categories because not all interior angles are acute and none of the interior angles are obtuse. Therefore the categories the author defined in argument two do not hold for all triangles and could be considered as not general. However, the line of reasoning the author used to prove the theorem for the two categories of triangles is valid and it would hold for right-angled triangles. The main issue with this argument is that it is not general based on the categorization the author used for covering the cases of possible triangles. This argument would be general if the author included a third category of all right-angled triangles. Nonetheless if students selected this argument as valid, I paid attention to their discussions of the generality to determine if they noticed the omission of right-angled triangles. Although the task did not entail students noticing the omission of right-angled triangles, the author of the textbook considered this approach as a valid proof of the theorem.

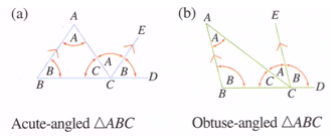
The third argument considered all possible triangles. I constructed this proof using the hints provided in another textbook, Chandler, Smith, Ali, Layne, and Mothersill (2008). The authors did not prove the exterior angle theorem but suggested that it can be proven using the straight-line theorem. The latter theorem claims that the sum of angles on a straight line is 180º. Therefore, I used the angles on a straight-line theorem and algebraic operations to prove the theorem is true for any triangle. Argument three also represented the fourth level of Balacheff

taxonomy of proof (i.e., thought experiment). Unlike argument two, this argument was general because it included all possible triangles. Arguments two and three differed in the theorems or axioms the authors used to construct the arguments.

The fourth argument was reflective of the second level of Balacheff's taxonomy of proof, the crucial experiment. I also authored this proof using a specific case of a *30-60-90°* triangle. In this proof, I demonstrated the validity of the theorem by means of a carefully selected example. The main tenet of this argument was my intention of showing that with the use of algebraic operations, the straight-line theorem and properties of the selected example, the theorem can be shown as true. Argument four was not general because I only considered a specific case of right-angled triangles. The argument did not include other types of right-angled triangles, acute-angled, or obtuse-angled triangles.

<table>
<tr><td colspan="2">Theorem: If any side of a triangle is extended then the exterior angle so formed is equal to the sum of the two remote interior angle measures</td></tr>
<tr><td>Argument 1: Naïve Empiricism

If I take a ruler and a pencil and draw any triangle say ΔABC shown below and extend the side $\overline{BC}$ to a point D. Now take my protractor and measure the exterior angle say BCD and then measure the interior angles A and B then you will observe that if you find the sum of the measures of the two angles A and B then that sum is equal to the measure of angle BCD. Once I show this for a few triangles then the theorem is true.</td><td>Argument 2: Thought Experiment

 
Acute-angled $\triangle ABC$ Obtuse-angled $\triangle ABC$
R. T. P. (Required to Prove) that if any side of a triangle is extended then the measure of the exterior angle so formed is equal to the sum of the measures of the two opposite interior angles.

To prove this we will need to consider all possible cases of triangles: acute-angled triangles and obtuse-angled triangle.

Considering any acute-angled triangle ABC and any obtuse-angled triangle ABC. Let us extend the side $\overline{BC}$ to D. Then draw $\overline{CE}$ parallel to $\overline{BA}$. Now $A\hat{C}E = B\hat{A}C = \hat{A}$ Because they are alternating angles resulting from the traversal AC cutting the</td></tr>
</table>

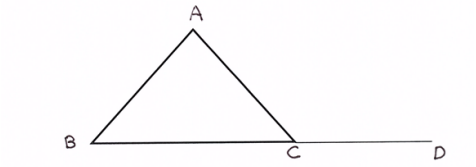
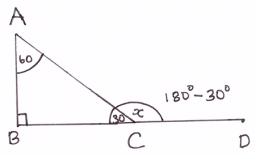
	two parallel lines BA and CE. And $E\hat{C}D = A\hat{B}C = \hat{B}$ Because they are corresponding angles resulting from the traversal BC cutting the parallel lines BA and CE. Therefore $A\hat{C}B + A\hat{C}E + E\hat{C}D = \hat{C} + \hat{A} + \hat{B} = 180^0$ (Angles on a straight line) And $A\hat{C}B = \hat{C}$ $A\hat{C}E + E\hat{C}D = \hat{A} + \hat{B}$ Subtracting $\hat{C}$ from both sides we have $A\hat{C}D = A\hat{C}E + E\hat{C}D = B\hat{A}C + A\hat{B}C = \hat{A} + \hat{B}$. Hence the theorem is proven. Q.E.D
Argument 3: Thought Experiment  Let *ABC* be any triangle, and extend $\overline{BC}$ beyond *C* to *D*, so that angle *ACD* is an exterior angle. $B\hat{A}C + A\hat{B}C + A\hat{C}B = 180^0$ [Sum of angle measures in a triangle] $A\hat{C}D + A\hat{C}B = 180^0$ [Sum of angle measures in a linear pair] $A\hat{C}B = 180^0 - A\hat{C}D$ [Subtraction property of equality] $B\hat{A}C + A\hat{B}C + 180^0 - A\hat{C}D = 180^0$ [Substitution property of equality] $B\hat{A}C + A\hat{B}C - A\hat{C}D = 0$ [Subtraction property of equality] $B\hat{A}C + A\hat{B}C = A\hat{C}D$ [Addition property of equality] Hence the result.	**Argument 4: Crucial Experiment**  Let us consider any 30-60-90 degree right-angled triangle where $\hat{A} = 60^0$, $\hat{B} = 90^0$ And $\hat{C} = 30^0$ Now let us extend $\overline{BC}$ to *D*. Let *x* represent the exterior angle *ACD*. Angles *C* and *x* are on a straight line so $\hat{C} + A\hat{C}D = 180^0$ [Sum of angles on a straight line] $\hat{C} + x = 180^0$ $30^0 + x = 180^0$ [so we have an equation where we use algebra and solve for *x*] $x = 180^0 - 30^0$ $x = 150^0$ And now $60^0 + 90^0 = 150^0 = x$ the exterior angle. Hence the result is proven.

Figure 4.3: The four arguments for students' selection as personal and best mark proof choice.

An Example of Coding for Generality and Validity

The following excerpt represents an example of how I coded students' responses for the generality and validity of a proof. When I asked the question, "Consider the following arguments for proving the theorem in Example 1. The theorem is taken from *Mathematics a complete course* (Toolsie, 2009). Which argument would be closest to what you would write? Please explain." Tia stated:

> Tia: Well, I will choose argument three because it holds for any triangle unlike argument one, which is measuring a few cases and argument, four which just looks at a specific case.

In the above excerpt I asked Tia about her personal choice for a valid proof argument. When Tia stated, "because it holds for any triangle" she seemed to base her selection on the generality of the argument. I coded this quotation as the first dimension code, students' criteria for validity. As Tia made the comparison of argument three with arguments one and four with regard to the generality, she demonstrated her understanding that a valid proof must be general. In particular when she referred to argument one's deficiency of only measuring a few cases and argument four presenting a specific case. Tia's criteria for identifying a valid argument seemed to align with the idea that generality determines the validity of a proof. As a result, I coded Tia's statement as pertaining to the perspective that a valid proof must be general according to dimension two of my taxonomy of codes.

Students' Conceptions of the Structure of a Valid Proof

In addition to determining the generality and validity of a proof, I was interested in the student's conceptions of the structure of a valid proof argument. I asked the question, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" My objective was to obtain through student's talk, their criteria for

identifying a valid argument. I paid attention to their descriptions of what they considered as a valid proof. As described in my explanation of the taxonomy of coding categories, I used the theoretical framework to determine the dimension two categories for the structure of a valid proof.

According to McCrone and Martin (2009), there are two characteristics of a valid proof in principle four of their theoretical framework. The first characteristic states that the statements in a proof must be ordered to conform to the laws of reasoning to provide a path from the hypothesis to the conclusion. From the first statement, I defined the coding category, the correct ordering of the necessary steps. By this I refer to the sequencing of the steps within a proof so that line of reasoning is maintained from the given statements to the conclusion.

The second characteristic in principle four of the theoretical framework states that the inherent logic determines the validity of a proof, not ritualistic aspects of the form in which it is presented. From this description, I defined the coding category inherent logic. By this I mean that the mathematical argument presented by the author of a proof, adhere to the laws of mathematical logic such as conditional implications, modus ponens, or hypothetical syllogism. The inherent logic of a proof highlights the line of reasoning that connects the steps in a proof.

Based on my initial analysis of the data I paid attention to any emerging themes from students' talk about validity. I was interested in deciphering what criteria students used to identify the validity of a proof. As a result of my preliminary analysis of the data, the following emerging themes were evident in their talk (a) the level of detail of the proof and (b) the use of previously proven results. I used these two themes as dimension two codes. The level of detail refers to the inclusion of all necessary steps that ensure the path of reasoning from the hypotheses to the conclusion. A proof is considered valid if it contains all the statements and

supporting reasons that explain why the conclusion follows from the hypotheses. The statements with supporting reasons could include other axioms, definitions, algebraic equations, or previously proven theorems. The aforementioned mathematical ideas make up the steps that will complete the proof argument. The use of previously proven theorems refers to the application of well-established results. By well-established results, I mean mathematical ideas that are accepted as true based on the axiomatic system of results or have been proven as valid. The validity of a theorem can be determined by the support provided through the existence of or relation to other mathematical results. I used the aforementioned codes to determine students' criteria for evaluating the validity of a proof.

CSEC Opportunities for Proof

To obtain students' opinions of the opportunities for proof in the CSEC examinations, I asked question 14 of the protocol. Figure 4.4 shows the respective question. As I explained in the introduction, the selected question represents a GCNE-type exercise that has become more prevalent in the recent examinations. I used this particular question to determine whether students identified the type of calculate and explain questions as an opportunity to develop a proof argument. I requested that students provide reasons to support their claims about whether an answer to the question was or was not a proof. Based on my preliminary analysis of the data, the emerging themes evident from students talk included, (a) calculate and explain problem is a proof and (b) explanations of why is a proof. I used these two initial dimension two categories to code the students' responses to question 14 of the protocol.

Consider the question in Example 2. This question is taken from the January 2014, CSEC Mathematics examination. In your opinion why or why not would you consider an answer to this question as a proof?

Example 2

In the figure below, **not drawn to scale**, the lines AQB and $CRPD$ are parallel ar
is a transversal. $PQ = PR$ and angle $QPR = 28°$.

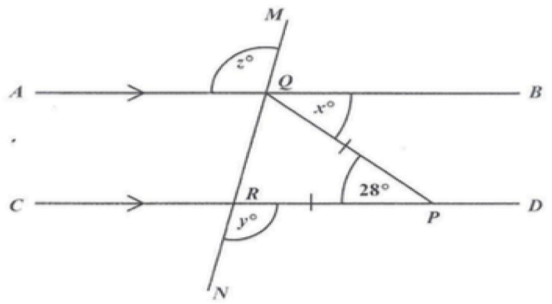


Figure 4.4: Question 14 of the interview protocol for investigating CSEC opportunities for proof.

Students' Conceptions of Teaching Practices of Proof

In this study, I also intended to use students as informants of classroom practices surrounding the teaching of proofs. I was interested in the teachers' textbook usage, the use of external examinations preparatory materials and any unique cultural practices teachers used during the instruction of proof. With regard to the textbook usage, I asked students questions about the content of the textbook as it relates to proof. My intention was to gather evidence of students' opinions about the opportunities for proof in their textbooks. I prompted students to talk about whether these opportunities were offered as individual classwork, group work, or as homework. I also inquired about how these opportunities were presented by their teachers. By the latter, I wished to learn about any classroom norms teachers established during their teaching

of proof. For this I considered whether teachers used open-discussions or group work. I paid attention to students' descriptions of any teaching practices during their discussions throughout the interview. I presented earlier, examples of the coding categories in Figure 4.2.

Reliability Coding

I conducted the reliability coding for the data with a fellow researcher. We randomly selected two students' interview data from each school. We used the taxonomy of coding categories and coded these six interviews for the categories (a) roles of proof in mathematics and school mathematics, (b) structure of a valid proof, (c) students and teachers criteria for generality and validity, (d) CSEC opportunities for proof, and (e) classroom practices for the teaching of proof. My decision to separate these into the respective categories was motivated by the conceptions I was interested in for answering the research questions. After conducting separate individual coding of the three interviews, we discussed any differences between our coding and any clarifications of coding categories. We obtained the following reliability coding for each of the above categories, (a) roles of proof in mathematics and school mathematics, 96%, (b) structure of a valid proof, 93%, (c) students and teachers criteria for generality and validity, 95%, (d) CSEC opportunities for proof, 98%, and (e) classroom practices for the teaching of proof, 97%.

Results

In this section, I present findings of my analysis of interviews with 21 students in Trinidad and Tobago. The findings are organized in sections answering each research question. These include findings related to students' conceptions of (a) the roles of proof in mathematics and school mathematics, (b) the structure and validity of a proof, and (c) opportunities for proof in the terminal examinations and curriculum. For each of the aforementioned sections, I present

representative quotes of students' talk during the interviews. With each quote I provide the context, including the interview questions or probing questions I asked during the semi-structured interviews. I discuss at the end of each quote, the prevalent themes that align with answering my research questions.

My first research question asked: *How do students in Trinidad and Tobago view the purposes of proof in secondary school mathematics?* To answer this question, I analyzed students' responses to questions three and five in the interview protocol (see Appendix F). Question three stated, in your opinion, why do you think mathematicians write proofs? In question five I asked, "In your opinion, could you provide reasons why you are taught proof in mathematics?" I specifically asked these questions to determine if there were any possible differences or similarities in students' perspectives about the roles of proof in the work of mathematicians and in their school mathematics experiences. In the following sections, I discuss the findings related to students' responses to each question.

Roles of proof in Mathematics

According to students' responses, the roles of proof in mathematics included verification, explanation, and systemization. Figure 4.4 presents a summary of students' conceptions of the roles of proof in Trinidad and Tobago. The verification role of a proof involves confirming that a statement or theorem is true. In the work of mathematicians, proof construction affords the opportunity for verifying that a claim or hypothesis is indeed true. The explanatory role of proof gives insight as to why a statement is true as opposed to confirming that it is true. The explanatory role of proof goes beyond demonstrating the truth of a statement and provides insight into the underlying mathematical relationships and concepts that make a statement or hypothesis true. In this role, proof is used to show the logical structure of mathematical ideas and

to make deductive lines of reasoning clear. The explanatory role of proof can help mathematicians discover patterns in mathematical relationships, and make further conjectures.

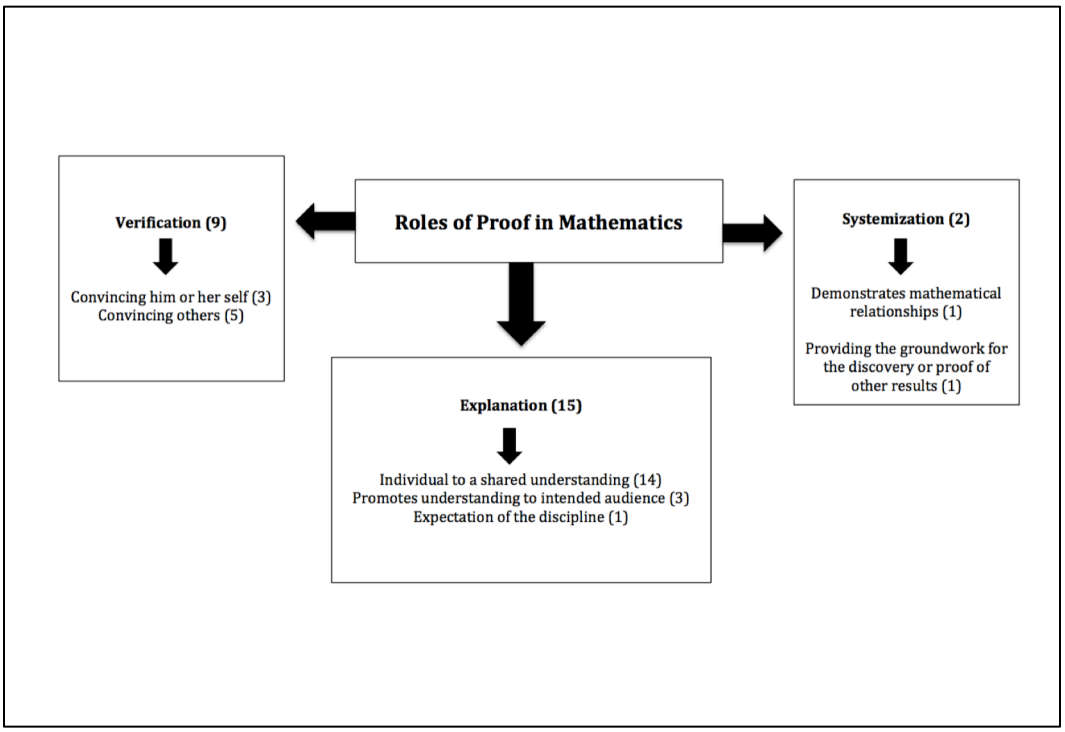


Figure 4.4: Students' conceptions of the roles of proof in mathematics.

Another role of proof in mathematics is the systematization of results into a deductive system of definitions, axioms, and theorems. In this role, proof helps mathematicians organize concepts, statements, definitions, axioms, and theorems from previously proven results. In Table 4.4, I present a summary of the meaning of each of these roles of proof students identified in mathematics. I also present counts of the number of students identifying each role. These counts do not indicate the number of times a student mentioned a role but the number of students who talked about a specific role.

Table 4.4

Roles of proof in Mathematics

Role of proof	Description	No. of Students
Verification	To verify that a statement of conjecture is true	9
Explanation	To give insight into why a statement is true	16
Systemization	To build an axiomatic system of results	2

In Table 4.5, I present the various roles of proof in mathematics that each student talked about during the interviews. As shown in Table 4.4 and Table 4.5, nine students identified the role of verification, with 15 students identifying the explanatory role of proof. Only two students talked about the systemization role of proof. Of the 21 students, 6 students talked about two of the three roles. Of these five students, four students discussed the roles of verification and explanation whereas one student talked about the roles of verification and systemization. Another student talked about the roles of explanation and systemization. In the next section I present representative quotes of students for each of the three roles of proof in mathematics.

Explanation Role of Proof in Mathematics

Of the 21 students, 16 talked about the explanatory role of proof in mathematics. Students stated in their responses that proof provides a justification of why a mathematical claim is true rather than showing that it is true. When I asked the question "In your opinion, why do mathematicians write proofs?" The following student, Jordan stated:

> Jordan: Well, in my opinion, mathematicians write proofs so that others can understand why their claims or theorems are true, like showing why whatever they are proving is true.

Table 4.5

Roles of Proof in Mathematics Identified per Student

School	Pseudonym	Roles of Proof in Mathematics		
		Verification	Explanation	Systemization
A	Cece		✓	
A	Faye		✓	
A	Jordan		✓	
A	Kelly		✓	
A	Mary	✓	✓	
A	Melissa		✓	
A	Tia		✓	
B	Danni		✓	
B	Kathy		✓	
B	Morgan		✓	
B	Noel	✓		
B	Oliver		✓	
B	Ruth	✓		
B	Taylor		✓	✓
C	Ava	✓	✓	
C	Dana	✓		
C	Delina		✓	
C	Ray	✓		✓
C	Zafiyah	✓	✓	
C	Shane	✓	✓	
C	Shanaz	✓		

Note: ✓ = A student identified this particular role of proof

Jordan identified the explanatory role of proof in a mathematicians work. She stated that by explaining why a mathematician's claim is true, others can understand. Jordan's phrase "like showing why" indicates the illuminating aspect of proof in presenting arguments that provide insight into why the claim is true as oppose to only verifying it is true. When Jordan stated, "so that others can understand why their claims or theorems are true" she seemed to identify the role

of explanation as a requirement for others to understand a mathematician' claims. Another student in their response to the same question stated:

> Ava: Mathematicians write proofs to explain. I mean that proof explains the things a lot easier for other people so they will understand why it is true.

Ava indicated that mathematicians write proofs to explain why a result is true. When Ava stated, "explains the things a lot easier for other people," she seemed to suggest that a proof provides an opportunity for a mathematician to explain to other persons why his or her claim is true. Ava voiced the opinion that in the explanatory role, a proof makes it "a lot easier" for others to understand how the mathematician's claim is true. When I probed further and asked Ava the question, "So what do you mean by explaining for other people?" Ava clarified:

> Ava: Well, what I mean is that when they [mathematicians] write their proofs, they are really helping others like other mathematicians and people who will read the proof see exactly how other mathematical concepts make up reasons supporting why the result is true rather than just showing it is true.

In this quote, Ava seemed to consider that "others" included other mathematicians and readers of the proof. Ava also asserted that proof in its explanatory role reveals mathematical relationships among "other mathematical concepts." According to Ava, these mathematical relationships help in the construction of logical reasons in support of why the result is true rather than demonstrating it is true. Of the 16 students who talked about the explanatory role of proof, 14 students talked about how the explanatory role of proof extends mathematical understanding from an individual to a shared understanding with others. Ava's quote is representative of this group of 14 students. According to these students, the role of explanation extends understanding from the mathematician, the author of the proof to other mathematicians and readers of the proof.

Among the students who talked about the explanatory role of proof in mathematics, three students talked about the intended audience going beyond mathematicians to include students

who will eventually use the proven result. For example, Melissa, in response to the question, "In your opinion, why do you think mathematicians write proofs?" stated:

> Melissa: I think they [mathematicians] write it to explain to other mathematicians and to students what is going on and why the thing [the mathematical claim] is true and this will help the students who will use it have a better understanding of why the thing they are proving is true.

In this quote, Melissa also identified the explanatory role of proof in a mathematicians work. When Melissa stated, "a better understanding of why the thing [the mathematical claim] they are proving is true," she seemed to consider the need to provide the intended audience with insight into why the mathematician's claim is indeed true. By intended audience, Melissa considered other mathematicians and students who will use the mathematical result being proven. Melissa seemed to view proof as serving the purpose of helping the intended audience understand the mathematical ideas proposed by the mathematician. When Melissa stated "what is going on and why the thing is true," she seemed to suggest that proofs are needed to promote understanding and also to help others pay attention to the internal logic within the proof's explanation. Melissa's quote is representative of the three students who seemed to consider not only other mathematicians but also students as consumers of the mathematician's claim.

Explanatory role of proof is an expectation in the discipline of mathematics. Among the group of students who talked about the explanatory role of proof, one student also discussed this role as a requirement of the discipline of mathematics. In particular Cece, held the perspective that providing explanations is an integral part of a mathematician's work and also adheres to an expectation of the discipline of mathematics. For example in her response to the question, "In your opinion, why do you think mathematicians write proofs?" Cece stated:

> Cece: Oh because mathematicians can't just make up a rule and give it to people and say here this is what it is, without explaining it. They have to prove why it is true and this explains how they got it and the reasons behind it.

In this quote, Cece highlights the explanatory role of proof by her use of the phrases "explain how you got it" and "the reason behind it." These phrases emphasize Cece's view that in making a claim, one must explain why the result is true and justify how it is applicable in the mathematical situation. When Cece stated, "mathematicians can't just make up a rule and give it to people and say here this is what it is, without explaining it," she highlighted the explanatory role of proof as an expectation of the discipline of mathematics. In this phrase, Cece described the requirement of providing explanations in support of one's claims, as a mathematical practice inherent in the work of mathematicians. According to Cece "they [mathematicians] have to" explain why their claims are valid. Cece views proof construction as an opportunity for fulfilling this obligation within the discipline of mathematics. This highlights the importance of proof in the work of mathematicians. Cece seemed to suggest that proof serves the role of explaining why a mathematician's claim is true and this is important because it adheres to a requirement of the discipline of mathematics.

Verification Role of Proof in Mathematics

In response to the question, "In your opinion, why do mathematicians write proofs?" Nine students identified the role of proof in verifying a mathematical claim. According to the students, proof serves the purpose of confirming whether a mathematical result is true. Students stated that the verification role of proof in a mathematician's work is a means of convincing themselves and others. Of the nine students, three students talked about verification as a means of mathematicians convincing themselves that their claims are indeed true. For example, Ava in her response to my question, "In your opinion, why do mathematicians write proofs?" stated:

> Ava: Well mathematicians really write their proofs to prove to themselves that their hypothesis is correct.

When Ava stated, “their hypothesis is correct.” she talked about mathematicians writing proofs to verify that their hypotheses are true. Ava’s use of the word hypothesis suggests that a mathematician’s claim has some measure of uncertainty and its validity still needs to be verified by the mathematician. When Ava stated, “to prove to themselves,” she seemed to suggest that the verification role of proof helps mathematicians convince themselves that their claim is true. This quote is representative of the three students who shared the opinion that the verification role of proof helps a mathematician convince him or herself. Among the nine students, another group of five students talked about the role of verification as a means of convincing others of the truth of a mathematical claim. For example, Mary in response to the same question stated:

> Mary: Well when they [mathematicians] write proofs it is really showing others a confirmation that their claims are true.

Mary described a proof as a “confirmation” meaning that in a mathematician’s work, a proof is a means of verifying that a claim is true. Mary also talked about the verification role of proof in convincing others. For example, when Mary used the phrase “really showing others,” it seemed to suggest her view that the mathematician is convincing others possibly within a mathematics community that his or her claim is true. This quote is representative of the five students who held the opinion that in verifying a result, a mathematician is attempting to convince others that his or her claim is indeed true.

Additionally within students’ talk about the verification role of proof in mathematics, they emphasized the necessity of this role in a mathematicians work. In particular, one student Noel, in his response to the question, “In your opinion, why do mathematicians write proofs?” responded:

> Noel: Well I think mathematicians write proofs to verify a result or claim is valid. You have to have proof to know that it is right or wrong.

Noel shared the opinion that mathematicians write proof "to verify" whether their claim is valid. Noel specified the verification of one's claim as an important aspect in the work of mathematicians when he stated, "you have to have proof to know that it [a mathematical claim] is right or wrong." Noel's phrase highlights the need for mathematicians to always verify their claims. For example, when Noel stated, "you have to have proof," it suggested his consideration of proof as a necessity for mathematicians to show the validity of their claims. Unlike the aforementioned groups of students who talked about the verification role, Noel did not specify the audience to whom the mathematician is verifying his result. However, when Noel stated, "You have to have proof to know that it is right or wrong" it could be inferred that verification could be for convincing him or herself or possibly others of the truth. This quote is representative of students' view that in verifying a statement, a mathematician fulfills the obligation of establishing the validity of their claims.

Systemization Role of Proof in Mathematics

In addition to the roles of verification and explanation, two students, Ray and Taylor, discussed the systemization role of proof in mathematics. I find this important because (a) students' identification of the systemization role distinguishes the views of the student population I interviewed for this study and (b) this role goes beyond the roles of verification and explanation, identified in the theoretical framework. Although in the literature there are many roles of proof, McCrone and Martin (2009) in their theoretical framework identified the roles of verification and explanation. As a result, of my use of their framework with a different population of students, my findings suggest an additional role of proof that adds to the theoretical underpinnings of the six principles of proof understanding in McCrone and Martin's (2009) framework.

The systemization role of proof in mathematics promotes the development of theorems from axioms. An axiom is any mathematical statement that serves as a starting point from which other statements are logically derived. In this case, the initial axioms provide the foundational concepts or building blocks that lead to the discovery and justification of other mathematical results. Within an axiomatic system, several mathematical theorems can be connected by common underlying axioms or results that have been proven as true. Students in their talk identified the role of proof in the systemization of mathematical results. In response to the question, "In your opinion, why do you think mathematicians write proofs?" Taylor responded:

> Taylor: Proofs are written to help provide the groundwork for other results you will meet later on. For example, like the theorem on the interior angle sum in a polygon with n sides, you will have to use earlier proven results such as the interior sum in a triangle is 180 degrees and there are *n-2* triangles in any polygon with n sides to prove later on that in a polygon with n sides you have interior angles adding up to 180 degrees multiplied by *n-2*. Additionally, I think the proofs highlight the importance of underlying mathematics that explains why it is true and connected to later mathematics.

As Taylor stated, "provide the ground work for other results later on," she suggested that there exists an axiomatic system of results in mathematics. By this she referred to the notion that in mathematics, several mathematical theorems can be developed because of other definitions, axioms or proven results. For example, Taylor refers to the theorem, which states that the sum of interior angles in an n-sided polygon is equal to $(n-2)\times 180°$. This result is based on proving and then using another theorem, which states that the sum of interior angles in a triangle is 180º. The proof of the latter theorem is based on two underlying axioms. The first is the sum of angles on a straight line adds up to 180º. The second is that when a line crosses two parallel lines they make equal angles (i.e., corresponding angles) with the two lines. Therefore the proof of the sum of the interior angles of a triangle can be constructed from these two fundamental axioms. Taylor

seems to suggest that the proven theorem and its underlying axioms provide the "groundwork" to prove the interior angles sum theorem for an *n*-sided polygon.

Taylor also identified that the proven fact "there are *n-2* triangles in any polygon with n sides" coupled with the previously proven theorem for a triangle, provide the foundational concepts needed to prove the interior angle sum theorem for any *n*-sided polygon. According to Taylor, one can deduce that the interior sum of the n-sided polygon is $(n-2)\times 180°$. Taylor mentioned this example to demonstrate how the axiomatic system is based on deducing new results from previously proven results and their underlying axioms.

Taylor also explained how the proof of a theorem emphasizes the importance of the underlying results used to prove a mathematical result. For example, Taylor made this point when she talked about how the proof of the sum of interior angles in a triangle illuminates the importance of the theorem and its underlying axioms within the axiomatic system that eventually deduces the interior angles sum theorem for of an *n*-sided polygon. The development of a proof argument helps build the foundation for the development of other resulting theorems within the unique axiomatic structure of Geometry. Furthermore Taylor's latter statement "connections to later mathematics" suggested that proving a result is integral to making connections to the development of later theorems within the axiomatic structure of Geometry. Taylor also discussed the role of proof in explaining why a result may be true. This quotation highlights the opinion that proof builds the axiomatic structure of content areas such as Geometry and promotes the explanation of why a result may be true.

Another student, in response to my question, "Why do mathematicians write proofs?" Ray stated:

Ray: To back up themselves to show they are right and really this is important because how mathematics topics are usually connected, he may need to verify that his theorem is right so that he can use it prove other results later on."

In his response, Ray firstly discussed the verification role of proof. When Ray stated, "to back up themselves to show they are right," he seemed to suggest that mathematicians write proofs to verify or "back up" that their claims are true. Ray's reference to backing up or supporting also suggested his consideration of the need for mathematicians to ascertain that their claims are valid. According to Ray, verifying the validity of one's claims is necessary for establishing the systemization of mathematical results. For example, when Ray stated, "he [a mathematician] can use it to prove other results later on" it suggested his opinion that the proof of one result leads to the development of other connected results. This refers to the axiomatic system of mathematics, whereby Ray affirms with his statement "mathematics topics are usually connected." The latter statement demonstrated Ray's understanding that there exists an axiomatic system of results in mathematics. Ray's quote demonstrates the opinion that the verification of the result affords the mathematician to be able to use this result to develop other results

Although Ray does not provide an example like Taylor, his understanding of the systemization role of proof is made concrete when he talks about how the proof of one result may lead to the possible discovery or justification of other results later on in the mathematician's work. Overall, Taylor thinks that proofs help highlight the connections of mathematical concepts and also how a proven result provides the foundation for developing the axiomatic nature of mathematics. Both students demonstrated their understanding of the systemization role of proof in mathematics. Ray and Taylor voiced their opinion that in mathematics, proof highlights the relationship between mathematical results.

Roles of proof in School Mathematics

In their responses, students identified two roles of proof in school mathematics. By school mathematics, I refer to the proof opportunities in school during the teaching of various secondary school mathematics topics. Students stated the following roles (a) promoting understanding and (b) appreciation of mathematics. Figure 4.6 shows the various roles of proof students identified in school mathematics. The number is parentheses represent the number of students who talked about a specific role of proof.

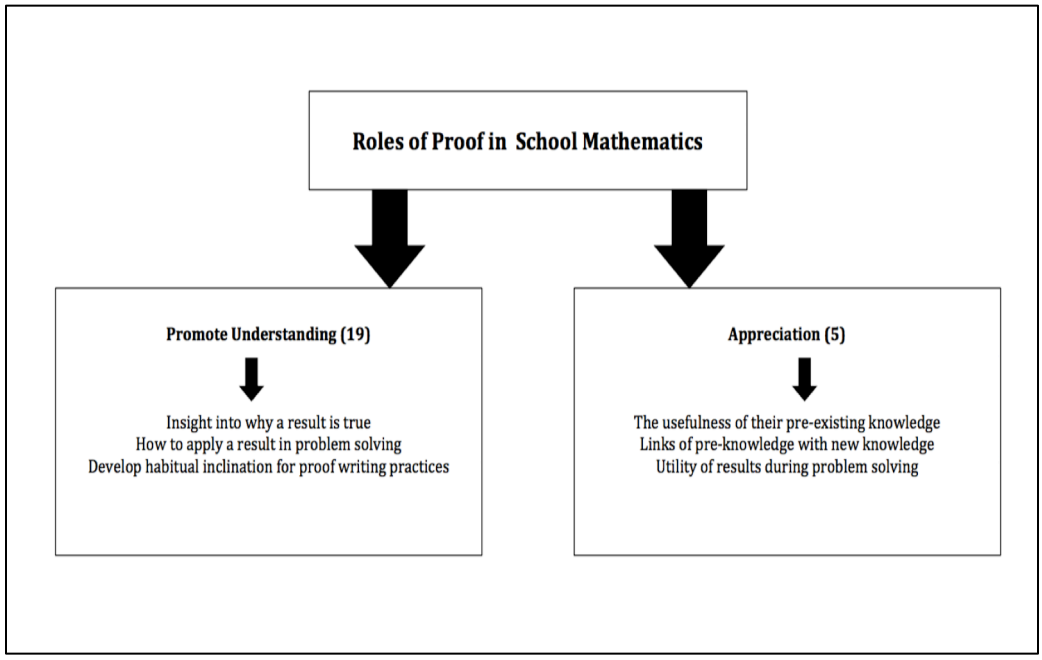


Figure 4.6: Students' conceptions of the roles of proof in school mathematics.

The promotion of understanding aligns with the explanatory role of proof in mathematics. My analysis of students' responses suggested that in school mathematics, the teaching of proof helps students understand why a theorem or result is true. Although this role is similar to the role of explanation in mathematics, students specifically talked about proof providing insight or a deeper conceptual understanding of why a result is true. Students suggested that this insight

helps them understand the mathematics they learn. Students spoke about the elucidation of underlying relationships between mathematical results when a proof explains why a result is true.

With regard to the explanatory role of proof in mathematics, students seemed to suggest that proof helps a mathematician extend his or her understanding to a wider mathematics community. This interpretation differs to students' descriptions of the promotion of understanding because the latter directly relates to their individual understanding of mathematics. As a result, I choose to keep these two labels and meanings as separate perspectives of students in this study.

The second role of proof evident in students' responses was appreciation of mathematics. In this role, proof allows students to value the usefulness of other related mathematical concepts when showing why a result is true. This role is an interesting finding because it demonstrates a unique conception of proof held by students in Trinidad and Tobago. I discuss the findings pertaining to this rule in a subsequent section. In Table 4.6, I present a summary of the description of these roles and a listing of the students who identified the aforementioned roles of proof in school mathematics. As shown in Table 4.6, 19 students talked about the role of promoting understanding in school mathematics. Of the 21 students, 5 students identified the role of appreciation. Of these five students, three also identified the role of promoting understanding.

Promoting Understanding in School Mathematics

In response to the question "In your opinion, could you provide reasons why you are taught proof in mathematics?" Several students talked about how the teaching of proof promotes understanding. Of the 21 students, 19 students talked about this role of proof in their school mathematics experiences. According to the students, three main themes emerged in their discussions of this role. Students claimed that through the promotion of understanding students

can develop (a) insight into why a theorem is true, (b) knowledge of the utility of a proven result, and (c) habitual inclinations for their own proof writing practices.

Table 4.6

Roles of Proof in School Mathematics Identified per Student

School	Pseudonym	Roles of Proof in School Mathematics	
		Promote Understanding	Appreciation
A	Cece	✓	
A	Faye	✓	✓
A	Jordan	✓	
A	Kelly	✓	✓
A	Mary	✓	
A	Melissa	✓	✓
A	Tia	✓	
B	Danni		✓
B	Kathy	✓	
B	Morgan	✓	
B	Noel	✓	
B	Oliver	✓	
B	Ruth	✓	
B	Taylor		✓
C	Ava	✓	
C	Dana	✓	
C	Delina	✓	
C	Ray	✓	
C	Zafiyah	✓	
C	Shane	✓	
C	Shanaz	✓	

Note: ✓ = A student identified this particular role of proof.

The first theme addressed the opinion of students who believed that the teaching of proof promotes understanding that helps them see how a result originated and why it is true. The second theme referred to students' opinions about how understanding why the result is true allows them to effectively use the proven result to solve problems. The third theme attended to

students' opinions of how the promotion of understanding leads them to see the importance of writing proofs that also promote understanding to others. Within this final theme, students stated that their teacher's demonstration of a proof helps them develop similar proof writing habits that allow readers to have the same understanding they experienced as a reader of their teacher's proof. It should be noted that various students' discussions ascribed to more than one of the aforementioned themes. In the following sections, I present representative quotes of each of these themes.

Insight into why a result is true. Of the 21 students, 19 discussed the promotion of understanding as a role of proof in school mathematics. In their discussions, all 19 students explained that through the promotion of understanding they gained insight into why a result is true. This insight pertains to students seeing how the internal logical deduction of the proof explains why the result is true. The following student in response to the question, "In your opinion, could you provide reasons why you are taught proof in mathematics?" stated:

> Kathy: Basically the teaching of proof really ensures that the students understands clearly why a result is true and by this I mean well you see how the argument flows and leads from the hypotheses to what you want to prove.

According to Kathy, proof in school mathematics "ensures" the promotion of understanding through the logical argument that supports why a mathematical result is true. Kathy's use of the word "clearly" suggested her view that a proof elucidates the necessary statements with their supporting reasons that establishes the validity of a mathematical result. When Kathy stated, "you see how the argument flows," she seemed to suggest that the proof's internal logic facilitates understanding why the proven result is true. Kathy seemed to consider the internal logic as the path of reasoning connecting the "hypotheses" (i.e., the given statements) to the

conclusion. Kathy's quote is representative of the view that proof in school mathematics allows students to understand the internal logic that explains why a result is true.

Another student, talked about another perspective of the promotion of insight into why a result is true. Jordan shared her opinion of the reasons for teaching proof in school mathematics. During a later portion of the interview, when we were discussing Jordan's teacher showing proofs to the theorems in their textbooks, I asked the follow up question, "Why do you think your teacher shows you the proof to the theorems in your textbook?" Jordan explained:

> Jordan: Uhm well the proofs are taught because it is good to understand where everything came from, like the background of how the mathematics rules and theorems we learn come about. I think understanding this background allow you to see why the rules and theorems are true.

Jordan's response is based on her reasons for why her teacher would usually show the proofs of theorems from their textbooks. Jordan explained that the promotion of understanding is a role of proof in her school mathematical experiences. According to Jordan, the promotion of understanding refers to explaining how the mathematician came up with the mathematical claim or result. For example, when Jordan stated, "to understand where everything came from" she seemed to refer to the proof's argument highlighting "how the mathematical rules and theorems" can be deduced from other mathematical results. Jordan seemed to hold the view that proofs highlight the connections between mathematical concepts. Jordan considered the connected concepts as the "background" (i.e., explanation) of how a mathematical result was deduced. When Jordan stated that this is "good" she seemed to highlight the intellectual need of students' understanding the mathematical results they encounter in their school experiences. As Jordan stated, "I think understanding this background allow you to see why the rules and theorems are true" she seemed to suggest that understanding how a mathematical result came about explains why the result is true. This latter statement emphasizes that proof provides insight into how a

mathematical result was developed and why the result is true. Jordan's quote is representative of the opinion that in school mathematics, proofs help students understand how a mathematical result was developed.

Another student Cece also talked about the role of proof in promoting understanding in school mathematics. When I asked the question "In your opinion, could you provide reasons why you are taught proof in mathematics?" Cece responded:

> Cece: In my math class, my teacher does not just give us a theorem. She starts from the beginning and shows us the entire proof so that when we actually start questions we'll understand what we're doing and understand that it makes full sense knowing why the result holds.

Cece explained how her teacher would usually spend time showing the "entire proof." By this she seemed to refer to the teachers inclusion of all steps linking the hypotheses to the conclusion of the proof. When Cece made the statement "so that it makes full sense knowing why" it suggested her view that the purpose of her teacher explaining each step in the proof is for promoting insight into why the theorem is true. Cece highlighted that the understanding promoted through the proof's elucidation of why the argument holds is helpful to students. Cece clarified that the proof helps students understand that "it [the proof] makes full sense" meaning the internal logic of the proof. The internal logic seemed to include the statements and supporting reasons that make up the steps in the "entire proof." When Cece stated "so that when we actually start questions we'll understand what we're doing" she talked about how the promotion of understanding is useful when applying the proven result in questions. The latter quote also demonstrated her understanding of the intellectual need of the reasons why the result is true when doing problems. Cece's quote is important because it demonstrated a student who discussed the first theme of promoting understanding and the second theme about the utility of proven results.

The promotion of understanding informs students of the utility of a proven result. Students claimed that the promotion of understanding allows them to see the usefulness of the result being proven. Students connected this usefulness to the solving of problems in school and applying the result during exams. When I asked the question "In your opinion, could you provide reasons why you are taught proof in mathematics?" Faye stated:

> Faye: To get a better understanding of what actually happens and explain why the result or theorem is true because if I get stuck somewhere along the way in a problem I can think back to how you got it in the first place and this helps me remember how the result came about and more importantly how to use it in the problem.

Faye explained that proof serves the purpose of promoting understanding in school mathematics. For example, Faye suggested by her statement "to get a better understanding" that proof promotes insight into "what actually happens." The latter statement seemed to suggest that proof affords students the opportunity to gain insight into the logical statements that elucidate why "the result or theorem" is valid. Faye elaborated further to explain the importance of gaining an understanding of why a theorem is true. Faye claimed that by understanding the internal logic within the proof of a result, she could effectively apply it during problem solving. In particular, when she is unable to recall the actual theorem. Faye claimed that during these instances, she can "think back" or recall the insight gained through the proof's explanation. Faye believes that the insight helps her recall the connection of the theorem to other mathematical concepts. As a result, Faye asserted that "more importantly" because of the insight, she remembered how to apply the theorem in solving the problem. Faye voiced the opinion that in school mathematics, the insight gained in understanding a proof could possibly extend to a future recollection and application of the proven theorem. This student's views suggest that understanding the logical arguments that support why a theorem is true helps students see the utility of the proven theorem and underlying axioms when solving problems.

Another student also talked about how proofs promote understanding that nurtures students' future application of the proven theorem. Melissa in her response to the question, "In your opinion, could you provide reasons why you are taught proof in mathematics?" indicated:

> Melissa: Because it [the proof] will give you a better understanding of why. For example why the Pythagoras theorem is true, somehow this is better than just learning off [memorizing] the theorem. When you see the proof is makes you understand why it is true and this will help you in exams to apply the result.

In her response Melissa asserted that proof provides "a better understanding of why" a theorem is true. In particular, Melissa highlighted that the teaching of the proof of Pythagoras' Theorem allowed her to understand why the theorem is true. In fact, Melissa sees the importance of gaining this understanding. According to Melissa, understanding why the theorem holds is "better," meaning more useful in an exam rather than memorizing the formula.

Melissa voiced the opinion that the teaching of the proof of theorems promotes conceptual understanding that seems more useful when applying the proven theorem. The conceptual understanding refers to Melissa's ability to reason effectively during an exam setting to efficiently apply Pythagoras' theorem to solve problems. The conceptual understanding that Melissa gained through the proof's explanation seemed to enhance her proficiency in applying the theorem accurately and efficiently. Melissa's quote highlights the view that the role of proof in promoting understanding leads to students' comprehension of mathematics concepts, which enhances their procedural fluency and ability to solve problems.

The promotion of understanding leads to developing proof writing habits. Students claimed that the promotion of understanding helps them develop habitual inclinations towards their own proof writing skills. In response to the question, "In your opinion, could you provide reasons why you are taught proof in mathematics?" Shane commented:

Shane: To help you understand why a theorem or formula is true and this helps you see the need to always defend your statements and explain your thinking in your proofs so others could understand why they are valid.

Shane explained that proofs in school mathematics helps students understand why a theorem or formula is true. Shane suggested that the promotion of understanding allows students to see the need for also writing proofs that promote understanding. Shane's quote seemed to suggest that by understanding why a theorem is valid, students see the intellectual necessity of developing the habitual inclinations "to always defend your statements" when writing their own proofs. Shane also stated that understanding also leads to developing the disposition of always explaining one's thinking and supporting one's arguments. Shane seemed to think that these inherited proof writing habits are important in promoting understanding to others who read his proofs. The promotion of understanding seems to initiate habitual inclinations that develop Shane' diligence in writing his own proofs, which also promote understanding.

Another student, Mary, discussed how the understanding she gained through her teacher's proofs of theorems (a) impacted her future application of the theorems when writing her own proofs, and (b) led to her adopting proof writing habits that also produce proofs that promote understanding why a result holds. When I asked Mary the question, "In your opinion, could you provide reasons why you are taught proof in mathematics?" she stated:

Mary: Proofs really help in understanding mathematics well. I mean like for example in a proof our teacher showed us for how two triangles are congruent, she explained why they were congruent. She used SAS [Side-Angle-Side postulate] to support why the triangles are congruent. I felt that understanding how to use those axioms to explain why is important because it helped me get in the habit of explaining how when writing proofs to show that two angles are congruent.

When Mary stated, "proofs really help in understanding mathematics well" she expressed the view that proofs are taught to promote understanding of mathematical concepts. Mary made her assertions by recalling a teaching episode when she gained insight into how to prove that the

congruency of a pair of triangles. Mary recalled that her teacher showed proof arguments that explained why two given triangles were congruent. Mary believed that the proof elucidated the relevance of the *Side-Angle-Side* (SAS) postulate in constructing arguments that prove the congruency of a pair of triangles. Mary commented that understanding the proof's explanation of why the triangles were congruent was important for two reasons. Firstly, she was able to see the intellectual need of the congruency postulates when proving congruency. Secondly, she developed the habitual inclination (i.e., "get in the habit") to write proofs that effectively apply the theorems to prove why a pair of triangles is congruent. Mary used the example of proving congruency of two triangles to further elaborate her view that the teaching of proof highlights how to apply one's mathematical knowledge to explain why a result is true. Mary and Shane shared the opinion that through their teacher's demonstration of proofs, which promote understanding, they developed the habit of writing proofs that also promote understanding.

Appreciation Role of Proof in School Mathematics

Of the 21 students I interviewed, 5 students talked about the role of appreciation. According to these students, the teaching of proof allows them to appreciate the usefulness of the underlying axioms and other mathematical results used to construct a proof. For example, when I asked, "In your opinion, could you provide reasons why you are taught proof in mathematics?" Danni explained:

> Danni: I think Miss really shows us the proof even though it is not required, to help us appreciate the mathematics you are learning like where it came from and how to use it to solve problems.

Danni held the opinion that although proofs are "not required" her teacher presented the proofs to help the students appreciate mathematics. According to Danni, this appreciation entails knowing the relationships between the mathematical concepts her teacher uses to prove a

theorem. The appreciation also consists of knowing how to effectively use the theorem to solve problems. Danni holds the view that knowing "where it [the theorem] came from" meaning the mathematical concepts that deduce the theorem, helps in appreciating the utility of the theorem in problem solving.

Another student, Faye, held a similar view to Danni. When I asked the question, "In your opinion, could you provide reasons why you are taught proof in mathematics?" she responded:

> Faye: So we can have an appreciation for the actual formulas that we have well you know like we could appreciate how the formula relates to other things we learned.

In her response, Faye talked about how the teaching of proof motivates her to see the value of "the actual formulas" she is learning in mathematics. According to Faye this value entails her understanding how the formula "relates to other things we learned." By this, she seemed to consider how a mathematical result is connected to its underlying axioms and previously proven results, she encountered in her mathematical experiences. Faye voiced the opinion that proof serves the role of motivating students to appreciate how a formula or theorem relates to their previous mathematical knowledge. This is important for students to possibly systematize their pre-existing knowledge with the new knowledge attained through the teaching of proof.

Another opinion students voiced when discussing the role of appreciation in school mathematics concerned the personal gratification students received during the teaching of proof. Taylor in response to the question, "In your opinion, could you provide reasons why you are taught proof in mathematics?" stated:

> Taylor: To help us understand and appreciate the mathematics we are doing cause if she [the teacher] does not show us the proof for a result we will just write it down and not really understand the purpose of it and how to connect it to other stuff and we would really be excited about it.

Taylor explained that proofs are taught to promote understanding and appreciation of the mathematics "we [students] are doing." Taylor explained that if her teacher does not share the proof of a result with the students they would not conceptually understand the result. This conceptual understand entails knowing the "purpose of it [the theorem]" and how it fits into the axiomatic system of mathematical results. When Taylor stated, "we [students] would really be excited about it," it seemed to suggest that there is a measure of personal gratification when a student sees the value of a mathematical result in their school mathematical experiences.

Finally another student made the connection that through appreciating the usefulness of related mathematical concepts, students are equipped to use them efficiently in problem solving. For example, Kelly in response to the question, "In your opinion, could you provide reasons why you are taught proof in mathematics?" asserted:

> Kelly: For understanding and secondly students can appreciate why the result is true and the purpose of the result meaning why it is useful. To know what they are doing and why she's applying a certain formula… and how to use it in a particular way to answer a question.

Kelly explained that besides the promotion of understanding, proof allows students to appreciate (a) the reasons supporting why the result is true and (b) the utility of the result for solving problems. According to Kelly the aforementioned implications of the role of appreciation on students learning help develop students' metacognitive skills. When Kelly stated, "to know what they [students] are doing," she seemed to refer to students' reflecting on their mathematical thinking as it relates to the mathematical result being proven. This statement also extends to students' reflecting on how to apply the proven result during problem solving. Kelly's opinion suggests that through the role of appreciation, students become aware of their own thinking processes about how to effectively apply a mathematical result during problem solving.

The fifth student, similar to the previous opinion talked about how their teachers' actions when writing a proof, exemplified the role of appreciation. In response to same question I asked the other four students, Melissa stated:

> Melissa: Well, the teaching of proof helps us to really value the usefulness of mathematics when you see how Miss apply some other results to prove something.

Melissa asserted that through her teacher's application of "some other results" to prove a theorem, she appreciated the utility of these results. By other results, Melissa seemed to refer to axioms, definitions, and other previously proven theorems that help support the new theorem she was learning. Melissa's appreciation consisted of a proof illuminating the usefulness of her pre-existing mathematical knowledge in establishing new knowledge. Melissa attributes the illumination to her teacher's effective application of the pre-existing knowledge to construct a proof. The "value" Melissa talked about seems to refer to the appreciation of the connection between pre-existing mathematical knowledge and the new knowledge she gained through the teaching of a proof.

Overall, these five students demonstrated through their opinions that the appreciation of mathematics is an important component to their learning in school. Each student articulated how the role of appreciation motivates an awareness of the usefulness of their pre-existing knowledge and the application of a new result. In particular, Taylor and Kelly emphasized that their appreciation of a new result allows them to see the connections to their previous knowledge. Melissa emphasized that through her teacher's application of previous knowledge when writing a proof, she valued the usefulness of her previous knowledge. As a result Melissa noticed the links between her previous knowledge with the new knowledge she learned. Faye and Danni asserted that through the appreciation of a mathematical formula, they became aware of its' utility during

problem solving. Both students felt that the proof of the formula helped them see the value of the formula when solving problems. These students also believed that the proof helped them become aware of how to apply the proven formula efficiently during problem solving.

Students' Conceptions of Proof Structure, Validity and Generality

My second research question stated: *What are students' conceptions about the structure, generality, and validity of a proof?* To gather students' opinions about the structure of valid proof arguments I asked students, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" This question was important because I expected students to provide their opinions about the characteristics of what they considered as a valid and convincing proof. In particular, by using the words convincing and correct, I intended to prompt students to express their own thinking's about the structure of a valid proof argument. I also intended that students had the full autonomy to share their own perspectives, hence my stating that the argument must be convincing or correct to them. In cases where students needed to clarify their descriptions of valid proofs, I asked follow up questions to allow them to clearly explain their ideas. Additionally, I gathered evidence of students' criteria for a valid proof by asking question nine of the interview protocol. In question nine I asked students to select from a four pre-written arguments, a proof that best represents what they would write. The prompt in this question afforded another opportunity for students to further clarify their opinions about the validity and generality of a proof. As a result, I paid attention to students' talk to determine any consonances or differences with their criteria when responding to the two questions. In cases where there were similarities, I considered their response as a prevailing conception of the structure of a valid proof argument.

Within students' descriptions of the characteristics of a valid convincing proof, I paid attention to characteristics that pertained to the structure of the proof argument. By this I mean how the argument is set up with regard to the nature of the statements, the steps, and the logical flow of the argument. According to the students, they considered the following attributes of the structure of a convincing and valid proof, (a) correct ordering of the steps, (b) inherent logic, (c) level of detail (i.e., the inclusion of all required steps), and (d) use of previously proven results. Figure 4.7 shows the characteristics of a valid proof the students identified in their talk. The number of students who talked about each characteristic is shown in the parentheses.

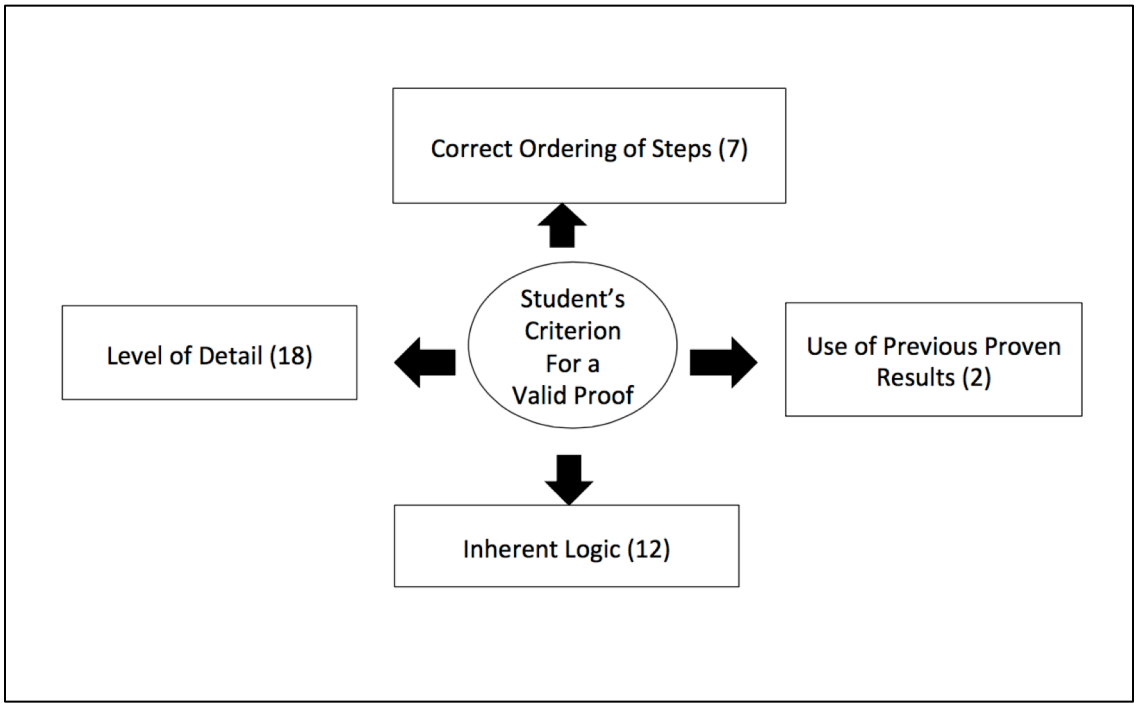


Figure 4.7: Students' conceptions of the characteristics of a valid proof.

The two first characteristics align with the statements about the structure of proof in principle four of the theoretical framework. According to McCrone and Martin (2009), the correct ordering of the steps refers to the sequencing of the statements with accompanying reasons in a proof. The sequencing must be accurate so that the line of reasoning from the

hypotheses to the conclusion is maintained. The inherent logic pertains to the line of reasoning that connects the steps in the proof. The line of reasoning should adhere to the laws of mathematical logic such as conditional implications, modus ponens, or hypothetical syllogism. The third characteristic, the level of detail, was an emerging theme evident in students' talk. The level of detail refers to the inclusion of all the necessary steps that support the logical path from the hypotheses to the conclusion. I will refer to the level of detail and the inclusion of all necessary steps interchangeably throughout my discussion of the findings related to this characteristic. This characteristic goes beyond those stated in the theoretical framework. The final characteristic pertains to the application of previously proven results to develop an argument that justifies the validity of a mathematical result. This characteristic also goes beyond those stated in the theoretical framework. In Table 4.7, I present the various characteristics of a valid proof structure each student identified during their talk.

Of the 21 students, 4 discussed three of the four characteristics, with two of these students from School C and one each from School B and C. Ten students discussed two of the characteristics, seven talked about one of the four characteristic. In the next sections I discuss each of these characteristics in detail. I present representative quotes and provide the context and implications inferred by students.

Level of Detail

Of the 21 students, 18 of them discussed the level of detail as a characteristic of a valid proof argument. By the level of detail, students seemed to refer to the inclusion of all the necessary steps. Students indicated that along with determining the validity, the inclusion of all steps promote understanding to readers of the proof.

Table 4.7

Characteristics of a Valid Proof

School	Names	Characteristics of a Valid Proof			
		Correct Ordering of Steps	Level of Detail	Inherent Logic	Use Previously Proven Results
A	Cece		✓	✓	
A	Faye		✓	✓	
A	Jordan		✓	✓	✓
A	Kelly				✓
A	Mary		✓	✓	
A	Melissa		✓		
A	Tia		✓	✓	
B	Danni		✓	✓	
B	Kathy		✓		
B	Morgan		✓		
B	Noel	✓	✓	✓	
B	Oliver	✓	✓		
B	Raegan		✓		
B	Taylor		✓		
C	Ava	✓		✓	
C	Dana	✓	✓		
C	Delina		✓	✓	
C	Ray	✓	✓	✓	
C	Zafiyah		✓		
C	Shane	✓	✓	✓	
C	Sean	✓		✓	

Note: ✓ = A student identified this particular characteristic of a valid proof

Students also defined what they meant by steps in a proof. Students described the steps as a combination of mathematical statements with accompanying reasons to support them. For example, in response to the question, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" Melissa responded:

> Melissa: Well a valid proof must have all the details I really mean the proof must have all the steps needed to prove the results. For example in a proof, a step has the

> statements like the sum of two angles *A* and *B* is *180º* with the reason, because the angles are on a straight line. You need both parts because if you leave out the reason someone will not understand how the statement fits into what you are saying and if you leave out an entire step it might disrupt the flow of the argument and also lead to some persons not understanding.

According to Melissa, "all details" refer to all the necessary steps for articulating a valid proof argument. Melissa defined a step in a proof as a statement and the accompanying reason for the statement. Melissa used the example of the statement, "the sum of two angles *A* and *B* is *180º*," and the reason "because the angles are on a straight line." This step is based on the theorem, which states that angles on a straight line add up to *180º*. Melissa highlighted that both components of a step must be included in a proof. Melissa held the opinion that if the reason is omitted from a step then a reader will be unable to understand why the statement is viable in the argument. When Melissa claimed that if the author omits a step it may "disrupt the flow of the argument," she seemed to refer to breaking the logical path of reasoning connecting the hypothesis and the conclusion of the argument. Melissa voiced the opinion that any break or disruption in the line of reasoning would lead to some readers not understanding the argument in the proof. Ten other students shared Melissa's definition of a step when discussing the inclusion of all steps as a characteristic of a valid proof.

Students articulated that the level of detail of a proof is determined by the inclusion of all required steps that ensures the validity of the proof. Students discussed the inclusion of all steps as an important requirement for validity by providing reasons for considering this requirement. Among the 18 students who talked about the level of detail, several students provided reasons why this characteristic determines validity. Students stated the following reasons (a) promotes understanding to all readers, (b) promotes individual understanding, and (c) ensures all readers follow the line of reasoning. The aforementioned reasons were evident in students talk about the

level of detail in a valid proof. My analysis suggested that students' reasons seemed to be determined by how they positioned themselves as either readers or authors of a proof. The first and second reason seemed motivated by students considering themselves as readers of the proof. The distinction between these two categories is that students either focused on all readers or solely on themselves. Whereas the third reason seemed motivated by students positioning themselves as the authors of a proof. I explain these reasons further in the following section. In Table 4.8, I present the reasons identified by students in their discussions about the level of detail characteristic.

Table 4.8

Reasons Why Level of Detail Determines Validity

School	Pseudonym	Reasons why level of detail determines validity			
		Promote understanding to all readers	Promote individual understanding	Ensure all readers follow the line of reasoning	No Reason
A	Cece			✓	
A	Faye		✓		
A	Jordan		✓		
A	Mary	✓			
A	Melissa	✓			
A	Tia	✓			
B	Danni				✓
B	Kathy	✓			
B	Morgan			✓	
B	Noel				✓
B	Oliver				✓
B	Raegan				✓
B	Taylor	✓			
C	Dana	✓			
C	Delina				✓
C	Ray		✓		
C	Zafiyah			✓	
C	Shane		✓		

Note: ✓ = A student identified this particular reason.

Of the 18 students, 6 provided the reason of promoting understanding to readers of the proof. Four students stated that the inclusion of all steps promoted individual understanding. Three students talked about how the inclusion of all steps ensures that any reader can follow the path of reasoning from the hypotheses to the inclusion. Five students did not provide reasons why the level of detail determines validity.

The level of detail promotes understanding to all readers. Of the 18 students who talked about the level of detail as a characteristic of a valid proof, 6 students stated that the level of detail promotes understanding to all readers. By level of detail, I refer to the inclusion of all the steps deemed necessary for supporting the validity of the proof. The six students explained that the inclusion of all required steps promote understanding to all readers of the proof. These six students positioned themselves as readers of a proof. In response to my question, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" Taylor responded:

> Taylor: Well it depends on how detailed it [the proof] is. I mean that in a valid proof, all the steps must be there so that everyone could follow from the start to the conclusion. Well what I mean is that when all the steps are there in the proof they explain why the conclusion is valid and this makes it understandable and convincing to all persons reading it.

When Taylor stated, "it [validity] depends on how detailed it [the proof] is" she seemed to consider the level of detail as a requirement of a valid proof. When Taylor stated that in a valid proof "all the steps must be there," she seemed to hold the view that the inclusion of all statements determines the validity of a proof. This later quote clarified Taylor's definition of the level of detail. Taylor also voiced the opinion that the inclusion of all the steps promotes understanding to all readers as they follow the connections from the hypotheses to the conclusion. For example, when Taylor stated, "so that everyone could follow from the start

[hypotheses] to the conclusion," she seemed to voice the opinion that the inclusion of all the steps is necessary for highlighting the path of reasoning that explains why the conclusion is viable from the given statements in the hypotheses. According to Taylor, the authors' inclusion of all the steps helps all readers understand why the argument is valid and also convinces them. Taylor's quote is representative of the six students who provided the aforementioned reason why the inclusion of all steps determines validity of a proof.

The level of detail promotes individual understanding. The promotion of individual understanding was another subtheme within students talk about the inclusion of all steps in a valid proof. As shown in Table 4.8, four students provided this reason for their consideration of the level of detail as a requirement for validity. In response to the question, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" Faye stated:

> Faye: Well to me in a valid proof they [authors] need to have all the steps because if they skip something I will get a little bit confused and I would not understand why the conclusion is valid.

Faye explained that she considered a valid proof as one that includes all steps. When Faye stated, "because if they skip something I will get a little bit confused" she asserted that the omission of a required step would lead to her not understanding why the proof is valid. The latter phrase also provides evidence of Faye positioning herself as a reader of the proof. Faye voiced the opinion that the purpose of including all steps is to promote individual understanding to her as a reader. Faye stated that this understanding entails explaining why the conclusion of the proof is indeed "valid" meaning true.

Another student who shared Faye's view also talked about individual understanding due to the inclusion of all steps. In response to my question, "What other characteristics do you look for to determine that a proof is correct or convincing to you?" Jordan explained:

> Jordan: Well all the steps must be included; they [the authors] can't leave out stuff because that will make sure I understand their argument. This really helps convince me that it is valid.

Jordan responded to my follow up question after giving her initial response about the characteristics of a valid argument. Jordan answered the follow up question and indicated that the inclusion of all steps is a characteristic of a convincing valid proof. For example, when Jordan stated, "all the steps must be included," she affirmed her opinion that including all the steps determines the validity of a proof. When Jordan indicated, "they [the authors] can't leave out stuff" she seemed to suggest that it is mandatory that the authors of a proof include all required steps. Jordan gave the reason for this requirement as to "make sure I understand" the presented argument. By this latter phrase, Jordan highlighted that the inclusion of all statements and supporting reasons facilitate her understanding of the reasoning path the author presented in their argument. The latter phrase also indicated Jordan positioning herself as a reader of the proof. Jordan's quote is representative of the four students who shared the opinion that the inclusion of all steps promotes individual understanding to a reader of the proof. According to these students, it seemed that individual understanding leads to convincing them that the proof is acceptable.

The level of detail ensures all readers follow the line of reasoning. Among the 18 students, three students talked about how the inclusion of all steps ensures that all readers follow the line of reasoning in the proof. For example, when I asked Morgan, "What would you look for

to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" she explained

> Morgan: Well a valid proof must have all the steps worked out so that it is clear for everybody. Because if a necessary step is left out then someone may get stuck there and don't follow why the proof is valid. All the steps must be there to ensure everyone is convinced that it is valid.

Morgan stated that the inclusion of all steps add to the clarity of the argument presented in a proof. For example, when she indicated "so that it is clear for everybody," she seemed to voice the opinion that a valid proof argument must efficiently convey through all its' steps an argument that promotes a shared understanding among its' readers. According to Morgan, the shared understanding depends on having "all the steps worked out." The latter phrase suggested her consideration of the level of detail represented by the inclusion of supporting statements with accompanying reasons or calculations.

Morgan also explained the importance of including all necessary steps. According to Morgan, the omission of a step prohibits readers from understanding beyond that step in the proof. For example, when she used the phrase "get stuck there," she seemed to point out that each step helps readers to follow the connections to previous and subsequent steps in the proof. The omission of an essential step could prevent readers from understanding the line of reasoning beyond that step to the conclusion of the argument. Morgan shared the opinion of the three students who talked about how the omission of a step can preclude readers from understanding the entire argument in the proof.

Ordering of the Steps

Among the 21 students I interviewed, 7 stated that the steps of a valid proof must be in a particular order. For example, Oliver in response to the question, "What would you look for to

determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" explained

> Oliver: To me a valid proof would have clear arguments; well I mean all the required statements with reasons to back it up must be there in the correct order because it allows you to follow from the hypothesis to the conclusion.

Oliver seemed to equate "clear arguments" as those which contain all the necessary steps. When Oliver stated, "all the required statements with reasons to back it up," he seemed to clarify that all the necessary statements along with their supporting reasons determines validity. Oliver also asserted that in addition to including all the steps, the author must present them "in the correct order." This latter phrase, demonstrated Oliver's opinion that the order of the statements also determines validity of the proof. When Oliver stated, "it [the order] allows you to follow from the hypothesis to the conclusion," he seemed to suggest that a valid proof must have all steps in the correct order so as to maintain the line of reasoning connecting the given statements to the conclusion of the proof. Oliver's quote is representative of a case where I double coded a student's response for the level of detail and ordering of the steps.

Another student Ray talked about the necessity of having all the steps in the correct order. In response to the same question that prompted Oliver's response, he stated:

> Ray: To be convinced by a proof and accept it as valid, all the steps must be there in the correct order cause this will help others understand it. You [the author] cannot just write a proof and mix up everything cause there is a purpose for having all steps in a particular order because you want everyone to understand your explanation from the start to the end of why it [the conclusion] is true.

Ray indicated that a valid convincing argument must include all the steps. Ray goes further to clarify that these steps "must be there in the correct order." When Ray stated, " you [the author] cannot just write a proof and mix up everything," he seemed to further clarify the importance of ordering the statements as a requirement of writing a valid proof. Ray's emphasis on order is

supported by the reason that the order will promote understanding to "others," meaning other readers of the proof. When Ray stated "to understand your explanation from the start to the end of why it [the conclusion] is true" he asserted that the correct ordering of the required statements reveals the line of reasoning that explains how the hypothesis leads to the conclusion. Ray interpreted this path of reasoning as an explanation. According to Ray, this explanation seems to refer to a reasoned argument that justifies the validity of the conclusion. Ray's quote represents a case of double coding, where a student talked about the inclusion of all steps (i.e., the level of detail) and correct ordering of the steps.

Inherent Logic of a Proof

Of the 21 students, 12 students stated that the inherent logic of the argument determines the validity of a proof. By inherent logic, students meant the mathematical logic that allows the path of reasoning from the hypothesis to the conclusion. For example, when I asked Noel the question, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" he stated:

> Noel: Well for me a valid convincing proof, I look to see if every step follows sensibly or make sense mathematically. Well what I mean by that is that every sentence mathematically follows from the sentence before.

In his response, Noel seemed to indicate that the inherent logic, which supports the ordering of the steps, determines the validity of a proof. When Noel gave the requirement that in a valid proof, each step "follows sensibly or make sense mathematically," he seemed to suggest that each step must logically connect to its preceding and subsequent steps. Noel seemed to suggest that the requirement "make sense mathematically" is determined by how each step enhances the internal logic of the argument.

Noel also voiced the opinion that the sequencing of each step is dependent on how the internal mathematical logic of the proof links the claims in each step. For example, when Noel stated, "every sentence mathematically follows from the sentence before," he seemed to suggest that beyond the sequencing of the steps, the steps must mathematically support the logical path of reasoning that confirms the validity of the proof.

Mary also talked specifically about the inherent logic confirming the validity of a proof. When I asked the question, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" she stated:

> Mary: I would look to see how clear the person explains their argument. I mean things like if it makes sense mathematically from the beginning to the end. I am convinced by a proof, which has sound logic because this makes the explanation much easier to follow.

Mary stated that a valid proof would have a clear explanation. According to Mary, the clarity is a result of the inherent logic of the argument in the proof. When Mary stated "makes sense mathematically from the beginning to the end" she seemed to suggest that the inherent logic of the argument must efficiently ensure that the path of reasoning connects the hypothesis to the conclusion. Mary also stated that she is convinced by a proof that has "sound logic." This description seems to suggest that the logical statements the author uses must adhere to the laws of mathematical logic and make sense. Mary seemed to consider that the laws of mathematical logic allows readers to understand the line of reasoning that explains why the conclusion is valid. Mary's quote is another example of students identifying the inherent logic as a determinant of a valid convincing proof.

In their discussions about the inherent logic, students provided reasons why they considered this characteristic as a requirement of a valid proof. Students suggested that the inherent logic ensures validity of a proof argument and promotes understanding to others within

a mathematical community. When I asked Shane, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument presented by someone in your group?" he stated:

> Shane: Well the proof must make sense to me and the other members of the group meaning that it follows logically from beginning to the end explaining why the conclusion can be made. For example if the proof has logical implications such as if *P* then *Q*, it must be true or must make sense and help support why the conclusion is valid.

Shane responded to the follow up question I asked during our discussions about the group work his teacher assigns when teaching proof. According to Shane, a valid proof must "make sense" meaning the proof logically explains why the result is true. Shane gave the condition that the proof must first be understandable to the community of learners in his group. Shane explained that making sense refers to the accuracy of the logical implications within the proof. He provided the example of the logical implication, which states If *P* then *Q*, where *P* and *Q* could represent mathematical statements in a proof. Shane used this example to help clarify his opinion that when logical statements are used in the proof they must be accurate and ensure the line of reasoning from the hypotheses to the conclusion.

Use of Previously Proven Results

During students' discussions about the characteristics of a valid convincing argument, two students talked about the use of previously proven results. Kelly and Jordan asserted that a valid proof in their opinion must possess this characteristic. When I asked Kelly the question, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" she explained:

> Kelly: Well I would look for how the other known theorems or mathematical knowledge is used to show how you got the result. This means looking at how the other well-known concepts and theorems we proved in class is used to help understand

> why the theorem we are proving holds. I really appreciate this and it convinces me it [the proof] is valid.

Kelly asserted that her criterion for a valid or convincing proof would be the use of other mathematical results when constructing the proof. Kelly clarified that these other mathematical results would include, "well-known concepts" and theorems previously proven in her class. By well-known concepts she seemed to refer to mathematical ideas that are taken as shared knowledge and also accepted as valid in her class. These concepts could possibly include axioms, and definitions. Kelly also considered theorems that were previously proven as true in her class. When Kelly stated, "I really appreciate this and it convinces me it [the proof] is valid," she seemed to value the efficient application of the previously proven results in constructing arguments that support the validity of another theorem. Kelly also considered that the use of this theorem would lead to convincing her that the proof is valid.

Jordan in response to the question, "What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?" explained:

> Jordan: Well I consider a proof as valid when the person writing it [the author] uses other results that were proven in class or shown in a textbook. I think that when they use these results to provide supporting reasons why, they show how different mathematical theorems are related and how some proven results lead to other results later on.

When Jordan stated that "[the author] uses other results that were proven in class or shown in a textbook" she suggested that the use of previously proven results determine validity of a proof. Jordan identified that the results can either be proven in her class or demonstrated in a textbook. The latter location seems to suggest her opinion of holding the textbook as an authority for establishing the validity of a mathematical result. When Jordan stated, "they use these results to provide supporting reasons why" she highlighted that the previously proven results are useful for providing supporting reasons that explain why the mathematical theorem is true. In her opinion,

this helps determine the validity of the mathematical theorem the author proved. Additionally as Jordan stated, "they show how different mathematical theorems are related" she shared her view that the use of the previously proven theorems elucidates the axiomatic system of mathematical results. Jordan clarified further as she stated, "how some proven results lead to other results later on." This latter phrase suggested that within the axiomatic system, the previously proven results contribute to the validation of later results. Jordan seemed to consider this systemization as a means of elucidating the relationships among different mathematical results.

Students' Conceptions of Generality of a Proof

To gather evidence of students' opinions about the generality of a proof, I provided an opportunity for students to select a valid proof from pre-written arguments during the interview. My objective was for students to have an opportunity to select an argument reflective of their own conceptions of validity and generality (i.e., their personal proof choice). I also intended for students to select an argument that reflects their teacher's criteria for a valid proof argument that would receive the highest mark (i.e., best mark proof choice). As I explained in the methods section, the four options of pre-written arguments varied in their generality. Arguments one and four were not general, whereas arguments two and three were general in nature. My analysis of students' personal proof choice suggested that students selected non-general and general arguments. In Table 4.9, I present the students' selections for their personal proof choice.

Of the 21 students, 16 selected non-general arguments. All 16 students selected argument 4 whereas none of them selected argument 1. Among the 16 students, 2 were from School A. All seven students, I interviewed from Schools B and C selected argument four as their personal proof choice. With regard to general arguments, five students selected these as their personal proof choice. All five students were from School A. Of these five students, three selected

argument two. Two students selected argument three. I present the findings related to these two categories in the following sections.

Table 4.9

Student's Selection of Personal Proof Arguments

School	Pseudonym	Students' Personal Proof Choice			
		Argument 1	Argument 2	Argument 3	Argument 4
A	Cece			✓	
A	Faye		✓		
A	Jordan			✓	
A	Kelly				✓
A	Mary		✓		
A	Melissa				✓
A	Tia		✓		
B	Danni				✓
B	Kathy				✓
B	Morgan				✓
B	Noel				✓
B	Oliver				✓
B	Raegan				✓
B	Taylor				✓
C	Ava				✓
C	Dana				✓
C	Delina				✓
C	Ray				✓
C	Zafiyah				✓
C	Shane				✓
C	Sean				✓

Note: ✓ = A student identified this argument as their personal proof choice.

Students' Selection of Non-general Arguments

Of the 21 students, 16 selected argument four as their choice of the proof that would best represent their own attempts. With regard to generality, none of the 16 students discussed the limitation of argument four using a specific case. Students' selections were based on (a) the low level of difficulty and use of algebraic operations with actual numbers (b) use of previously proven results, (c) inclusion of all steps (i.e., level of detail), and (d) inherent logic. In Table 4.10, I present the reasons for selecting the non-general argument per student.

Table 4.10

Reasons for Selecting Non-General Arguments

School	Pseudonym	Reasons for Non-General Argument Choice			
		Low level of difficulty	Use of previously proven results	Includes all required steps	Inherent logic
A	Kelly			✓	✓
A	Melissa		✓		
B	Danni	✓			
B	Kathy				
B	Morgan		✓		
B	Noel	✓			✓
B	Oliver	✓			
B	Raegan	✓		✓	
B	Taylor		✓		
C	Ava				
C	Dana				
C	Delina				
C	Ray	✓			
C	Zafiyah	✓			
C	Shane	✓		✓	
C	Sean	✓			

Note: ✓ = A student identified this particular reason.

As shown in Table 4.10, 16 students selected arguments four. Of these, eight based their selection on the low level of difficulty, with four students each from Schools B and C

respectively. Three students provided the reason of the use of previously proven results. Of these three students, one was from School A and two were from School B. None of the students at School C provided this reason. With regard to the reason of the inclusion of all required steps, three students provided this reason. One student from each school was in this group of students. Two students based their selection on the inherent logic of argument four. I present representative quotes for each of the aforementioned reasons in the following sections.

Reasons for Selecting Non-General Arguments

Of the 16 students who selected argument four, 8 students based their selection on the low level of difficulty. By this students seemed to refer to the authors use of algebraic operations with known values for a 30-60-90º right-angled triangle. Students considered this type of argument easier to follow as opposed to the use of the theorems and a generalized proof in arguments two and three.

For example, when I asked Zafiyah the question, "Which argument best represents one you would most likely write? Give me reasons why?" she responded:

> Zafiyah: Well I would actually go with argument four because I like how it uses a right angle example which has known angles that I can manipulate to get the outer angle of 150º and then add the 90 and 60 to give me the same 150. This proof is less difficult than two and three and one is definitely out because measuring is not proving. I would most likely write argument four.

Zafiyah's selection seemed to be based on the practicality of the triangle having known measurements. When Zafiyah stated "I can manipulate to get the outer angle of 150º and then add the 90 and 60 to give me the same 150" she seemed to value the usefulness of having known values that could be used with her algebraic knowledge to show the expected result. Zafiyah explained that knowing the interior angles, she would be able to firstly determine the exterior angle of the extended side of the triangle. Secondly, Zafiyah would be able to use the sum of the

other known interior angles (i.e., 60º and 90º to get the expected 150º). When Zafiyah stated, "This proof is less difficult than all the rest and I would most likely write that," she seemed to consider the use of a specific case as practical and appropriate for her mathematical ability.

Zafiyah used the level of difficulty as a criterion for not selecting arguments two and three. Additionally, when Zafiyah stated, "measuring is not proving," she acknowledge her belief that argument one is not a proof because using measurements of generated examples does not meet her requirement of a proof. Zafiyah's quote is representative of the eight students who selected argument four based on their perception of the level of difficulty.

Students also suggested that the inherent logic and the inclusion of all steps influenced their selection of argument four. For example, in providing reasons for selecting argument four as a personal proof choice, Kelly responded:

> Kelly: I would say four because it has all the steps there so I was able to understand it and also it seems a bit logical how the argument goes from the hypotheses to the conclusion that is proven.

Kelly based her selection on two reasons, (a) the inclusion of all steps and (b) the inherent logic. When Kelly stated, "because it has all the steps there so I was able to understand it," she suggested that the inclusion of all required steps led to her selection of argument four. Kelly also stated that the level of detail allowed her to understand the line of reasoning presented in the argument. As Kelly asserted "it seems a bit logical how the argument goes from the hypotheses to the conclusion" she seemed to value the inherent logic that facilitates the flow of the argument from the hypotheses to the conclusion. Although the argument was a particular case, Kelly based her selection on the flow of the argument along with the inclusion of all steps.

Of the 16 students who selected non-general arguments, 3 based their personal proof choice on the use of previously proven theorems. For example, when I asked Morgan, why would you select argument four as your personal choice?" She responded:

> Morgan: Argument 4 because it is simple and they gave you the angles to work with and here you use the sum of angles on a straight line is 180º, which we proved already, so you just subtract the 30º to get 150º. Also you have the right angle is 90º added to the 60º will give you 150º and will prove your point.

Morgan referenced "the sum of angles on a straight line theorem" as being integral in showing why the result holds. When Morgan stated, "you use the sum of angles on a straight line is 180 which we proved already" she asserted that the use of a result that is shared as valid supports the arguments presented in the proof. Morgan explained further how the author used specific measurements in the 30-60-90^0 triangles to demonstrate the validity of the theorem. Additionally it seemed that the given measurements helped in working out the answer and showing the two angles was indeed equal.

Students' Selection of General Arguments

Of the 21 students, 5 discussed the issue of generality as they made their selections of either arguments two or three for their personal proof choice. The students based their selections on the following reasons (a) holds for any triangle, (b) use of previously proven results, (c) limitations of the other arguments, and (d) teachers' previous assessment of student's proof writing. In Table 4.11, I present the aforementioned four reasons identified per student.

As shown in Table 4.11, all five students selected the general arguments because the authors considered all possible cases of triangles. Four of the five students provided the reason of the author using previously proven results as supporting reasons for justifying the theorem.

Table 4.11

Reasons for Students' Selection of General Arguments

School	Pseudonym	Reasons for General Argument Choice			
		Holds for any triangle	Use of previously proven results	Limitations of non-general arguments	Teacher's previous assessments of student's proof writing
A	Cece	✓			
A	Faye	✓	✓	✓	✓
A	Jordan	✓	✓	✓	
A	Mary	✓	✓		
A	Tia	✓	✓	✓	✓

Note: ✓ = A student identified this particular reason.

Three students eliminated the other proof options due to the limitations of (a) empirical reasoning and (b) non-generality. By empirical reasoning students meant the author's use of a few generated examples and measuring to prove the theorem as in argument one. Non-generality referred to the author focusing on a specific case as in argument four. Two of the five students based their selections on their teacher's previous assessment of their proof writing.

In response to my question, "Could you give reasons why you would select argument three?" Jordan stated:

> Jordan: I will choose argument three, because well first I'm going to extend the base of the triangle I would use to do the proof. I wouldn't think of a specific triangle like the right angle triangle. I will draw any triangle because that will cover all possible triangles and not just a few. I also like how other theorems we proved in class like the one about the interior sum in a triangle is used to prove this result, it really supports the claims you make in the proof.

Jordan explained that she would use the method the author of argument three employed to prove the result. Jordan stated that she would firstly extend the base of the triangle she would draw to start her proof. When Jordan stated, "I wouldn't think of a specific triangle like the right angle

triangle," she suggested her opinion of the limitation of using a specific case to write a proof that holds for all triangles. Jordan also identified this limitation as her reason for not selecting a non-general argument (i.e., arguments one and four).

When Jordan declared, "I will draw any triangle because that will cover all possible triangles and not just a few," she affirmed her preference for a proof that will hold for all triangles. Jordan's idea of using any triangle seems to be motivated by the requirement to have a proof that is general in nature. The phrase "will cover all possible triangles and not just a few," also demonstrated her understanding that argument one may be limited because it employs the use of only a few triangles drawn by the author to prove the theorem.

Additionally Jordan also selected argument three because it used theorems she had previously proven in class. According to Jordan, the use of the proven results helped in supporting the mathematical claims made in the proof. Jordan seemed to consider these theorems as the reasons that supported the statements in the steps of the argument. For example, when Jordan talked about "the interior sum in a triangle" she referenced the following step in argument three, $B\hat{A}C + A\hat{B}C + A\hat{C}B = 180^0$ [Sum of angle measures in a triangle]. Jordan valued the use of the theorem as a supporting reason for the sum of the three angles, *A*, *B*, and *C* in the triangle is 180^0 because this result may have been proven in her class. When Jordan mentioned this theorem she seemed to consider it as a shared valid result that is useful in supporting the claims about the angles in the triangle drawn in argument three.

Another student also discussed the limitation of selecting arguments one and four. Faye selected argument two as her option for the proof she would most likely write. Faye's selection was also based on her teacher's previous assessment of her proof writing skills. In her response

to the question, "Which argument best represents one you would most likely write? Give me reasons why?" she explained:

> Faye: I would definitely go with argument two because argument two you can see here that it holds for these different types of triangles, acute-angled and obtuse-angled triangles. This grouping should cover all triangles. I also find that unlike argument one which is measuring just a few cases and four which is a specific case, this one [argument two] seems to be for any triangle and Miss [her teacher] always say to write your proof so that is could hold for all cases that's why I would also go with this one also for my teachers option for best mark.

Faye identified that the classification of all triangles into acute-angled and obtuse-angled covered all possible triangles. Faye did not notice that right-angled triangles were not included in the author's classification. Nonetheless when Faye stated, "this grouping should cover all triangles," she seemed to understand the need for constructing a proof that is general in nature. Faye also confirmed this necessity for generality by adhering to her teacher's advice when she stated, "Miss [her teacher] always say to write your proof so that is could hold for all cases." The latter quote also suggested the possible influence of a teacher's expectations on students' conceptions of the generality of a proof.

Faye also voiced her opinion about the limitations about arguments one and four. For example, when Faye explained, "unlike argument one which is measuring just a few cases," she seemed to consider the limitations of using the measurements of a few generated empirical examples to prove a theorem that holds for all triangles. Additionally Faye identified argument four as "a specific case," thus suggesting her view that proving for a specific case of a right-angled triangle, does not extend to all cases. Faye was among the two students, whose selection was influenced by the their teacher's previous assessment of their proof writing skills. Faye and Tia's selection seemed to be influenced by their teachers' expectations for a valid proof. Therefore they selected the same argument as their best mark choice.

Tia, in response to the question, "Which argument would be closest to what you would write? Please explain," stated:

> Tia: I will go with argument two because it is general and not specific like argument four which only holds for that case of right angle triangles. Additionally you are using already well-known results to prove this result. For example, the theorem about transversal cutting a pair of parallel lines to get corresponding and alternating angles which are equal. In argument two they just apply what was proven already in class or a textbook to show the theorem holds, so I'll definitely go with argument two.

Tia explained that argument two is general in nature. Tia made a comparison of the generality of argument two and argument four. Tia stated that her selection of argument two was based on its generality unlike argument four which holds for only a specific case of a $30\text{-}60\text{-}90^0$ right-angled triangle. Tia also gave the reason that her selection was based on the author's use of other "well-known results." In particular she referenced the theorem about the angles formed when a transversal cuts across a pair of parallel lines. When Tia stated, "they just apply what was proven already in class or a textbook," she seemed to consider this "well-known" result as shared knowledge among an established authority such as textbook authors. Tia in response to my question, "Which argument do you suppose would receive the best mark from your teacher? Please explain," explicated:

> Tia: Argument two again, because as I said before for my choice, you're using theorems that have been proven before and it is general in nature and my teacher will value a proof which holds for all triangles.

Tia established that her teacher would "value" a proof argument that uses previously proven theorems and is general. When Tia stated, "my teacher will value a proof which holds for all triangles," she established that her teacher expected students to construct proofs that are general in nature. Tia also based her selection on the author's use of previously proven theorems. When Tia stated "because as I said before for my choice" she seemed to assert that she shared

the same proof requirements as her teacher. Tia's decision for her personal proof choice seemed to be influenced by her teachers' expectation for a valid proof.

Students' Selection of Best Mark Proof Choice

Table 4.12 presents each student's selection for the argument that would receive the highest mark from their teacher. Among the 21 students, 18 selected argument two as their best mark proof choice. Two students selected arguments three and one student selected argument four. In the following sections, I present representative quotes for the aforementioned reasons for students' selections of their best mark proof choice.

Table 4.12

Student's Selection of Best Mark Proof Arguments

School	Pseudonym	Student's Best Mark Choice			
		Argument 1	Argument 2	Argument 3	Argument 4
A	Cece		✓		
A	Faye		✓		
A	Jordan		✓		
A	Kelly		✓		
A	Mary			✓	
A	Melissa				✓
A	Tia		✓		
B	Danni		✓		
B	Kathy			✓	
B	Morgan		✓		
B	Noel		✓		
B	Oliver		✓		
B	Raegan		✓		
B	Taylor		✓		
C	Ava		✓		
C	Dana		✓		
C	Delina		✓		
C	Ray		✓		
C	Zafiyah		✓		
C	Shane		✓		
C	Sean		✓		

Note: ✓ = A student identified this argument as their best mark proof choice.

Reasons for Selecting Best Mark Proof Choice

The students selecting argument two discussed (a) the generality, (b) inclusion of all statements, (c) inherent logic, (d) the use of previously proven results, and (e) the level of difficulty as reasons for their selection. Of these 18 students, two selected argument two for their personal proof choice and best mark proof choice.

In response to the question, "Which argument do you suppose would receive the best mark from your teacher? Please explain." Raegan responded:

> Raegan: I will go with argument two because of the use of the alternating angles theorem, which will really help you show how the angles are equal even though they did not give you actual values for the angles. Also the statements are all there and it is easy to follow logically from the start to the finish.

Raegan's choice was determined by the use of other results to construct the proof of the theorem. Raegan stated the usefulness of the alternating angles theorem to help explain why the theorem is true. When Raegan stated, "the statements are all there and it is easy to follow mathematically from the start to the finish," she established additional requirements that guided her selection. Firstly, she suggested that her teacher's preference would be based on the inclusion of all required statements that facilitate the logical linking of the hypotheses to the conclusion. Secondly when she used the phrase " follow logically" she suggested that the inherent logic in the argument afforded a clear path of reasoning from the given statements to the conclusion of the proof.

Another student in response to the same question about the best mark proof choice, stated:

> Shane: For best mark from Sir [his teacher], I would go with argument two. Well first because it looks harder because it has more statements and uses other theorems we know like the Z angles [alternating angles] are equal, and in exams Sir gives

> more marks when you demonstrate that you can apply the theorems you learned in class to write a proof. It will show him that we understand the material.

As Shane stated, "because it looks harder" he seemed to firstly base his selection on the level of difficulty. When Shane declared, "because it has more statements" he seemed to associate the level of difficulty with the proof having more statements or possibly longer explanations. Shane also considered that the use of the geometric result about the formation of alternating angles when a transversal cuts across a pair of parallel lines would make the proof appealing to his teacher. The alternating angles theorem or "Z angles" theorem as he calls it, is a result that Shane seemed to consider as well known and accepted as valid. The use of this result facilitated the construction of a proof argument that will give him the best mark by his teacher. When Shane stated, "Sir gives more marks when you demonstrate you can apply the theorems" he seemed to suggest that his teacher valued students' ability to efficiently apply their mathematical knowledge. Shane's last phrase, "it will show him [the teacher] that we understand" suggested that the use of previous results is a means of formative assessment of students learning. As a result, Shane's selection was guided by the requirement of using the mathematical knowledge he acquired during his teacher's instruction.

Another student, in response to the question, "Which argument do you suppose would receive the best mark from your teacher? Please explain," explained:

> Sean: Well argument two because it is complex, I mean a lot more harder than the other arguments and this is so because it kind of holds for all triangles when they talk about for all acute-angled and obtuse-angled triangles and I also think because it uses other theorems we know are true such as the Z angles theorem and corresponding angles in the proof. So I think that's why Sir would give the most marks for that.

When Sean stated, "I mean a lot more harder than the other arguments" he affirmed that the level of mathematical difficulty of the proof guided his selection. He seemed to consider the level of

difficulty of argument two to be higher or more "complex" than the other available options. Sean provided two reasons for the higher level of difficulty. Firstly his statement, "it kind of holds for all triangles" suggested his view that the argument was general. Sean clarified this in his acknowledgement of the classification of triangles into acute-angled and obtuse-angled. Sean's use of the qualification of "kind of" somewhat indicated the possibility that he realized that the classification did not include right-angled triangles. However, I did not prompt him further to gather supporting evidence of this limitation in argument two.

The second reason for the high level of difficulty was the use of other theorems. When Sean declared, "because it uses other theorems we know are true," he asserted that the use of previously proven results support the level of difficulty that would be appealing to his teacher's grading scheme. Sean's quote demonstrated his opinion that the generality and use of other results are characteristics of a proof that would meet his teacher's expectation for a valid proof.

Among the students selecting the best mark proof choice, two students selected argument three. Mary and Kathy based their selection on the generality, and the inclusion of all steps. For example, in response to the question, "Which argument do you suppose would receive the best mark from your teacher? Please explain," Mary explained:

> Mary: Well I would go with argument three. Because it definitely holds for any triangle, also it has a lot more working because it includes all the steps. This is important because it explains clearly to your teacher so she knows exactly what you're thinking and your explanation of why the conclusion holds.

When Mary stated, "because it definitely holds for any triangle" she seemed to suggest that argument three was general in nature. Mary voiced the opinion that the generality of argument three guided her selection. Additionally as Mary stated, "it has a lot more working because it includes all the steps" she identified that the inclusion of all steps as another desired characteristic of her teacher's expectation for a valid proof. When Mary stated, "she [teacher]

knows exactly what you're thinking and your explanation of why the conclusion holds," she highlighted that the latter requirement promotes understanding to her teacher. Mary also explained that the clarity of the argument allows her teacher to understand the student's line of reasoning in explaining why the theorem is valid. This quote is representative of the reasons students provided for selecting argument three as their best mark choice. However, none of these students selected this argument also as their personal proof choice.

Of the 21 students, one student Melissa, selected argument four as her best mark proof choice. Melissa also selected this argument as her personal proof choice. I find this interesting because Melissa represents a unique characteristic of the student population in this study. When I asked Melissa the question, "Which argument do you suppose would receive the best mark from your teacher? Please explain," she explained:

> Melissa: I would also go with argument four because she [my teacher] would like how all the necessary steps are nicely laid out and using the angles on a straight line theorem we already proved in class. Also she always tell me that I need to write out all my statements and not assume that everyone would follow if a step is missing.

Melissa's selection was based on the inclusion of all steps and the use of previously proven theorems to construct argument four. When Melissa stated, "all the necessary steps are nicely laid out" she asserted that having all the required steps in a proof will appeal to her teacher's expectation. Melissa clarified the importance of adhering to this expectation when she stated, "I need to write out all my statements." Melissa provided the reason that her teacher would usual advise her to write all her statements. The latter statement also provided evidence of Melissa positioning herself as the author of a proof. According to Melissa her teacher cautioned that she should "not assume that everyone would follow if a step is missing." The latter quote suggested the teacher's influence on Melissa's development of her proof writing ability. The teacher's

previous assessments recommended that Melissa should include all steps in her proof so as to promote understanding to all readers. The teacher's advice also seemed to emphasize the importance of including every necessary step in a proof. Melissa seemed to value her teacher's expectation in her metacognitive development of her proof writing skills.

Melissa also highlighted that argument four contained other previously proven theorems. In particular, when Melissa referenced the theorem about the sum of angles on a straight line is 180º, she acknowledged the usefulness of this result in the proof of argument four. When Melissa talked about the theorem as one that was previously proven in her class, she seemed to consider the validity of the result as shared knowledge. Melissa's decision seemed to be influenced by the author's use of theorems proven by her teacher. Interestingly Melissa did not discuss the limitation of the generality of argument four. With regard to her selection of argument four as her personal proof choice, she also discussed the aforementioned requirement of including all steps.

Opportunities for Proof in CSEC Examination

In question 14 of the interview, I asked students' their opinions about whether a GCNE exercise would be considered a proof. In the introduction of this paper I highlighted the prevalence of calculate and explain type of questions in the CSEC mathematics examination. Given the high stakes of this exam in determining students' future engagement with higher-level mathematics, it is important to investigate whether these opportunities promote constructing proofs. It is also important to determine whether students considered the required informal explanations as potential opportunities for constructing a proof.

Of the 21 students in this study, all students claimed that a response to this question would qualify as a proof. Students provided the following reasons, (a) a clear explanation of why

your answer is true is a proof, (b) a proof is an explanation that supports your answer, and (c) an explanation that promotes insight into your thinking is a proof. In Table 4.14 I present a summary of the students who provided the aforementioned reasons.

Table 4.14

Reasons Why a GCNE is a Proof

School	Pseudonym	Reasons why a GCNE is a Proof		
		Clear explanation why	Proof is an Explanation	An explanation that promotes insight
A	Cece		✓	
A	Faye		✓	✓
A	Jordan	✓	✓	
A	Kelly			✓
A	Mary		✓	
A	Melissa		✓	
A	Tia	✓		
B	Danni	✓		
B	Kathy			
B	Morgan		✓	
B	Noel	✓		
B	Oliver	✓		
B	Raegan	✓		✓
B	Taylor		✓	
C	Ava			✓
C	Dana		✓	
C	Delina		✓	
C	Ray	✓		
C	Zafiyah	✓		
C	Shane	✓		✓
C	Sean	✓		

Note: ✓ = A student identified this particular reason.

When I asked Sean the question, "Consider the question in Example 2. This question is taken from the January 2014, CSEC Mathematics examination. In your opinion why or why not would you consider an answer to this question as a proof?" He responded

> Sean: Well thinking about it, the question specified that you must provide supporting reasons for your calculations. In this case, I really think they [the examiners] want you to give a clear explanation supporting why your answer came out to be that way. Well to me explaining why is a proof of your claim. I believe by explaining this the examiners will understand your thinking.

Sean reflected on the examiner's requirement of providing supporting reasons for calculations. When Sean stated, "I really think they [the examiners] want you to give a clear explanation supporting why your answer came out to be that way," he suggested that the examiners expected students to provide supporting reasons for the steps taken to calculate the unknown values. As Sean stated "to me explaining why is a proof of your claim" he voiced the opinion that providing an explanation of why a claim is valid, qualifies as a proof. Sean explained further the necessity for providing a clear explanation. For example, when he stated, "I believe by explaining the examiners will understand your thinking" Sean suggested that the examiners will understand the line of reasoning students use to compute the unknown values. This latter quote also demonstrated that Sean saw the intellectual need of explaining one's thinking when proving.

Students in their responses to question 14 stated that a proof is an explanation that supports your answer. Of the 21 students, 9 expressed this view. For example, when I asked Delina the question "Would you consider a response to this CSEC question a proof?" she stated:

> Delina: It is a proof because you give all the reasons why your answer is correct and this supports your answers like explaining why it is what it is. If you only put the answer you will get half or less of the marks. Also you have an opportunity to use your knowledge of the Geometry theorems you learn along with algebra to come up with supporting reasons and this is like writing a proof.

Delina explained that constructing a proof is equivalent to providing explanations that support the results of your computation. When Delina stated "If you only put the answer you will get half

or less of the marks" she seemed to highlight the necessity of adhering the examiner's expectation of supporting reasons for computations. Delina held the view that without these supporting reasons, the student will not receive full credit for the question. Furthermore, Delina considered the GCNE question as an opportunity to use pre-existing knowledge. Delina seemed to view the use of previously learned theorems along with algebraic operations to develop supporting explanations qualifies as a proof opportunity.

Another student, Kelly gave reasons that the exercise requires a clear explanation why and the explanation promotes insight. When I posed question 14 to Kelly she stated:

> Kelly: Well this question requires that you really explain clearly why you got the answers for the unknown values. To me this is what a proof does it clearly explains why a claim is true. Also when you provide the explanation you reveal to any one reading your proof the mathematical theorems that justify why your answer is the value you found. And again this is what a proof does it helps other see why. I recalled that my teacher always say, to write your answer so that the examiners can follow what you are saying

Kelly seems to consider the requirement of providing explanations similar to providing a proof. When Kelly stated "To me this is what a proof does it clearly explains why a claim is true" she asserted that a clear explanation of why a claim is valid is a proof. Kelly also suggested providing explanations promotes insight to readers of your proof. Kelly considered this insight as the elucidation of the mathematical knowledge that support why the resulting computation is valid. When Kelly stated "this is what a proof does it helps others see why" she suggested that the required explanation of one's calculations promotes insight to others reading the solution. Kelly seemed to consider that the insight allows "the examiners" to follow the line of reasoning that from the given info to the calculated values. Additionally by seeing the necessity to provide clear explanations that promote insight to the examiners, Kelly seemed to value the examiners' expectation of developing rationales that are equivalent to writing a proof.

Summary

Overall students in this study demonstrated an understanding of the various roles of proof and what constitutes a valid proof. Figure 4.8 summarizes students' conceptions of the roles of proof, structure, validity and generality of proof. Students considered the need for a mathematician to extend his individual knowledge to a shared understanding among other mathematicians and even to other consumers of the proof. In some cases, students articulated that the intended audience of the mathematicians work extended to students who will use the proven result.

Students considered the roles of verification, explanation and systemization in a mathematician's work. With regard to school mathematics students identified the roles of promoting understanding and appreciation. The latter role suggests that students see the intellectual value of the mathematical results their teacher proves during instructions. The role of appreciation also helped students see the utility of their pre-existing knowledge to prove new results they are learning.

Students also articulated various characteristics of valid proof arguments. Students highlighted the (a) the inclusion of all steps (i.e., level of detail) (b) correct ordering of steps (c) inherent logic, and (d) use of previously proven results as characteristics of a valid a convincing proof argument. Student's selection was based on them positioning themselves as either the author or reader of the proof argument. Throughout the interview, students had opportunities to state their opinions about the generality of a proof. These opinions were based on student's selection of a personal proof choice and best mark proof choice.

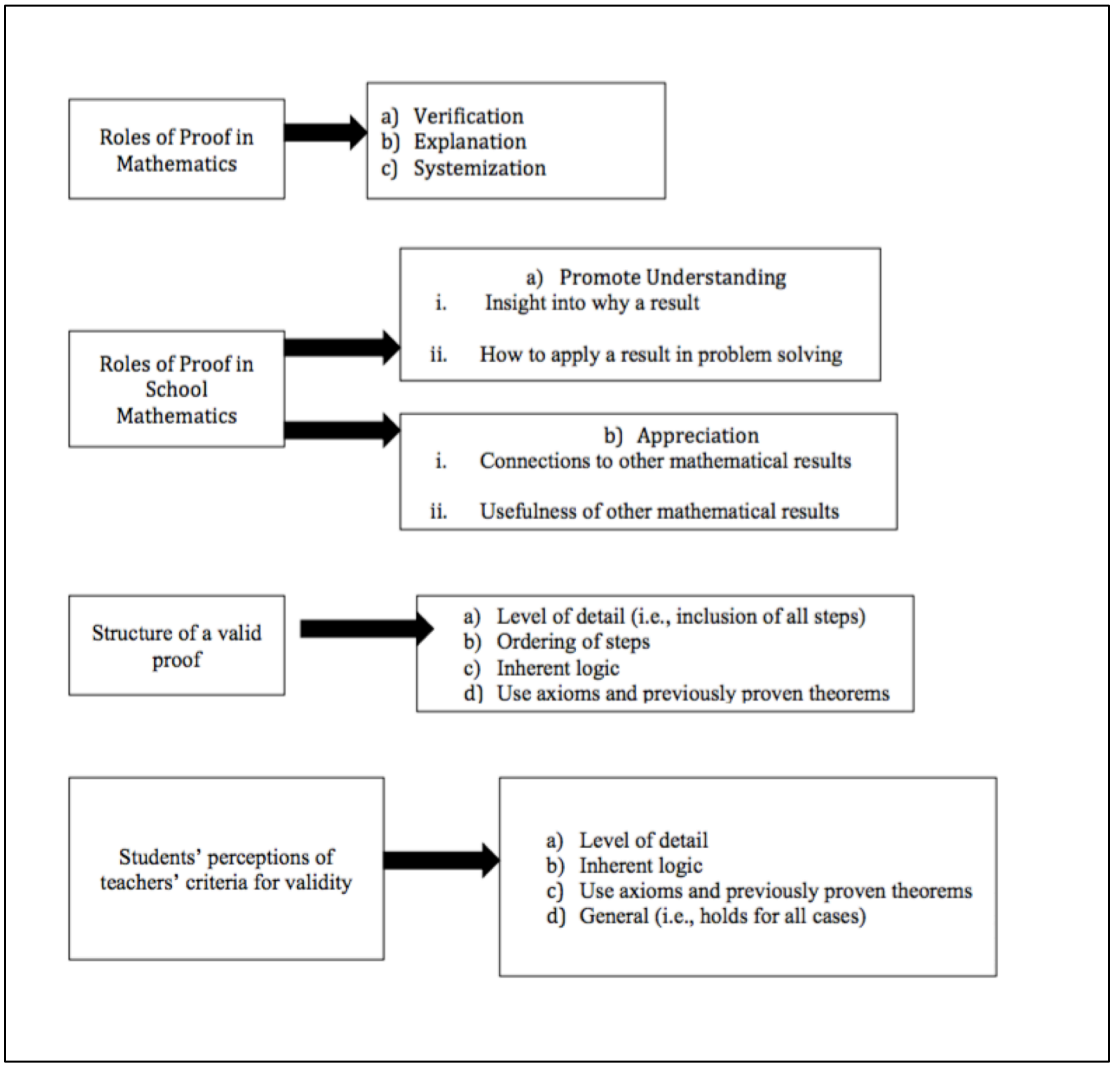


Figure 4.8: Summary of students' conceptions of proof.

Students provided reasons that suggested some correlation on the criterions for both selections. In particular, student's selection of the best mark choice indicated that teachers' expectations influenced students' criterion for the personal proof choice. For example the teachers' previous assessment of student's proof writing ability motivated students to select general arguments over the non-general options. This shows the correlation of the disciplines expectations on students' metacognitive development of their proof writing skills. Overall students considered the GCNE type exercises in CSEC examinations as opportunities for them to

construct proofs. Students understanding of the explanatory role of proof helped them see calculations with explanations as a proof opportunity. Students also asserted that an explanation is a proof. This suggests students' conceptions of what in their opinion constitutes a proof.

Discussion

In this study, my overarching question asked: *What are students' conceptions of proof in Trinidad and Tobago?* My inquiry was driven by my hypothesis that the recent reform in Trinidad and Tobago has implications for the mathematics content taught in schools and students learning about proof. Despite the calls for proof to take an important role in all students secondary school mathematical experiences, it is evident from the students' low performance on CSEC items that student in Trinidad and Tobago do have difficulties with proof. To understand this issue of students' low performance and the larger issues of the reform implications on students learning, I included this study in my dissertation. There are no existing studies examining students' conceptions of proof as it relates to the recent reform effort. Therefore my findings will inform this deficit in the existing discussions about mathematics education in Trinidad and Tobago.

Based on my analysis, the following main ideas were evident from students' conceptions of proof. First students viewed the teaching of proof as integral to them understanding the intellectual necessity and application of the mathematical results they learn in school. Second, students held the opinion that through a proofs' role of promoting insight, they were able to see the connections between their past and present mathematical knowledge. In particular, they felt that a proof elucidated the mathematical relationships between previously proven results and their current learning of concepts. The students also claimed that the elucidation of relationships emphasized the axiomatic system of results in Geometry. Third, the students talked extensively

about the explanatory role of proof. This included (a) the promotion of understanding why a result is valid rather than showing it is true (b) highlighting mathematical relationships between other previously proven results, and (c) explanation as a requirement of proof in the discipline of mathematics. Fourth, students held the perspective that in school mathematics, proof has the role of appreciation of mathematics. Students' talk about this role demonstrated their understanding of the intellectual necessity of the mathematical knowledge they acquire at school. Finally, students indicated that there exist several characteristics for a valid proof and some of these characteristics were influenced by their teachers and examiners expectations of what counts a valid proof. In the following sections I elaborate further about each of these high points in my analysis of the interview data.

Proof as a Methodological and Confirmative Tool

In this study, my first research question inquired about students understanding of the role of proof in mathematics and school mathematics. Several researchers in mathematics and mathematics education have shown that proof in a mathematician's work or in school has different purposes (e.g., Herbst & Brach, 2006; Lakatos, 1976; Rav, 1999; Thurston, 1995). Within a mathematician's work, proving is the activity of discovering and justifying one's claims. Proof is considered the result of such activity and demonstrates the mathematicians' line of reasoning in discovering and justifying his or her hypotheses. Proving is seen as the creative endeavor by which mathematical questions are studied and answered through the production of arguments that support and validate hypotheses (Herbst & Brach, 2006). A proof allows a mathematician to record the activity that culminates after pattern identification, making and testing conjectures, and generalization of claims. Additionally a proof can also be a catalyst for the further discovery of new knowledge (Thurston, 1995). Mathematicians use proof as a

"methodological tool" (Herbst & Brach, 2006, p. 75) to discover and substantiate mathematical knowledge.

However in school mathematics, proof is seen more as a confirmative tool whereby students examine the validity of theorems and claims they encounter or will use in their mathematics (Knuth, 2002). Proof is not used to discover new knowledge but as a tool for seeking confirmation of the validity of already proven and accepted results (Schoenfeld, 1994). My analysis suggests students' conceptions of the roles of proof align with these claims in the literature. According to students in this study, in school mathematics proof confirms already established result. The confirmation is needed for their verification of the validity of mathematical results. However beyond verification, students articulated that proof explains why and a result is true and allows them to see the intellectual need of the result in their mathematical experiences. According to the students, their respective teachers' demonstration of proof helps them see the value of the result being proven and helps them understand why the result is true. The latter requirement is important because the promotion of understanding helps students see the intellectual need for the result and also helps in their future application of the result.

Connecting Previous and New Mathematical Knowledge

Students also claimed that the promotion of insight afforded by a proof helps them see how the proven result is connected to other mathematical results. In particular students somewhat viewed proof as a bridge between their new knowledge and previous knowledge. This is important because it highlights unknowingly to some of the students the systemization role of proof. Only two students clearly articulated this role of proof in mathematics. None of the students explicitly talked about this role in their school mathematical experiences. However students implicitly spoke about this role in their discussions about proof elucidating the

connections of previously proven results or axioms to the new result their teacher introduced to the student. The students stated that their teachers' use of the previously established results to validate a newly introduced result helped them see the connections or mathematical relationships between their previous and new knowledge.

The systemization role helps in synthesizing students' mathematical knowledge into an organized structure of results. In fact, previous literature calls this the building of the axiomatic structure of results (Bell, 1976; Knuth, 2002). By axiomatic structure, I refer to the connected system of geometrical concepts, whereby a geometrical concept, postulate, or theorem forms the basis for discovering and proving other geometrical result. The introductory results (i.e., concepts, postulates or theorems) are either taken to be true or proven by a mathematician. Each of these established geometrical result forms a building block in the unique axiomatic structure of this domain of mathematics. In the teaching of Geometry, students are introduced to new geometrical concepts by confirming the existence and validity of earlier results within the axiomatic system of Geometry. I find this important because students' talk about this role of proof gives an indication of how students are taught Geometry in Trinidad and Tobago. Knuth (2002) noted that although in school mathematics, Geometry is taught according to an axiomatic system (i.e., Euclidean Geometry), it is uncertain that students are aware of the underlying axiomatic structure. Knuth further explained that students most likely consider that each geometric theorem is independent of each other even though they are taught within the Euclidean axiomatic system. However, the students in this study demonstrated to some extent, their understanding of the underlying axiomatic system of results in Geometry rather than the independently proven results. Based on students' answers in the interview it seems that the teaching of Geometry in Trinidad and Tobago does include opportunities for students to reflect

on Geometry from a meta-level. By this I mean that Geometry instruction helps some students become aware of the underlying axiomatic structure of Geometry in their secondary school experiences.

The Explanatory Role of Proof

Several authors have proposed roles of proof in the discipline of mathematics. Proof is central to communicating, verifying, discovering, explaining mathematical knowledge and systematizing mathematical statements in a mathematicians work (e.g., Bell, 1976; de Villiers, 1999; Hanna 1990, 1995, 2000; Schoenfeld, 1994). In a mathematicians work, students in this study considered the explanatory role as integral to promoting understanding. According to these students, the role of explanation extends understanding from the mathematician, the author of the proof to other mathematicians and readers of the proof. Mathematicians are interested in "more than whether a conjecture is correct, mathematicians want to know why it is correct" (Hersh, 1993, p. 390). In this study, 15 students talked about the explanatory role of proof, of these, 4 talked about how the explanatory role of proof extends mathematical understanding from an individual to a shared understanding with others. This is an important finding because it aligns with the reformer' advocacy for a social constructivist approach to the learning of mathematics in Trinidad and Tobago. Within a constructivist approach to teaching and learning, the more knowledgeable other extends his understanding to others within a mathematical community. In this case, the mathematician as the author of the proof explains why his claim is true. The explanation helps others gain an understanding of the mathematician's thinking processes that lead to the supporting arguments for his or her claim. Students' consideration of this suggests to some extent that they understand the need for a mathematician to communicate his knowledge to a level that it is shared among other mathematicians. It also suggests that the construction of

mathematical knowledge goes through the phase of individual to a community level whereby others appreciate the usefulness of the claim presented by the author.

The intellectual need of a mathematician's work with proof. Among the students who talked about the explanatory role of proof in mathematics, three students talked about the intended audience of consumers of the proof including students. The three students talked about the intended audience as going beyond mathematicians to including students who will use the proven result. In this group it seemed that students believed that mathematicians write proofs for the benefit of sharing knowledge with students who will eventually use the result. Herbst and Brach (2006) argue that although students are consumers of the knowledge produced by mathematicians they should have authentic experiences with the type of activities mathematicians engage in to create proofs of the results they use in school. Therefore students positioning themselves as direct consumers of mathematicians work can possibly help them value the intellectual value of the results they learn.

My analysis of students' responses suggested that students held the opinion that proof has a dual purpose in a mathematician's work. Firstly as a methodological tool that facilitates the discovery of mathematical results. In this capacity the proof a result confirms its validity and usefulness of the result by other mathematicians in their discovery of related mathematical results. These discoveries demonstrate the underlying axiomatic structure of mathematics. Secondly as a confirmative tool that convinces students of the validity of the results they will use in their school mathematical experiences. Students expressed the view that a mathematicians proof provides evidence of the validity of mathematical theorems and results they encounter in their school mathematics experiences. However, the discovery of some mathematical results may not be solely applicable for school mathematics but necessary for scientific discoveries and real

world applications. Nonetheless, by including themselves in the intended audience, students seemed to see the intellectual need of the work done by mathematicians.

Explanation as an Inherent Requirement of Mathematics

Among the group of students who talked about the role of explanatory role of proof, one student also discussed this role as a requirement of the discipline of mathematics. In particular Cece, held the perspective that providing explanations is an integral part of a mathematician's work and also adheres to an expectation of the discipline of mathematics. This view is important because it demonstrated a unique characteristic of the student population in this study. Although one student expressed this opinion, it is worth investigating in a future study whether other students held this view. The consideration of explanation as integral in a mathematician's work, places the emphasis on the latter stages of the processes of reasoning and proof. Stylianides (2009) explained that within a mathematician's work, the development of non-proof arguments to support conjectures, help explain why the conjectures are true. This stage goes beyond verifying or testing one's conjectures. In fact, the verification leads into the development of logical reasons that support the generalization of the conjecture. This development requires explanations of why a tested conjecture and the production of a reasoned argument or proof for their initial hypothesis.

Students' identification of the explanatory role of proof in a mathematician's work helps them see that in their own reasoning and proof processes, they need to similarly provide explanations for claims they make. This is an important part of the scaffolding needed to possibly push students to always develop non-proof and proof arguments to support their claims. In Trinidad and Tobago, the policy makers suggested that students should have opportunities to develop claims and provide supporting arguments for their claims (Republic of Trinidad and

Tobago, Ministry of Education, 2009). Students' understanding of the necessity or expectation of explaining why in the work of mathematicians can help them fulfill the policy maker's vision for authentic mathematical experiences for students. Mathematicians claim that the status of a proof is enhanced if it explains why a proposition is true as opposed to just confirming that it is true (e.g., Bell, 1976; Hanna, 1990; Hersh, 1993). Cece's views of proof agree with these researchers claims about proof as an inherent and important aspect in the work of mathematicians.

Students also identified the role of verification in a mathematicians work. Students seemed to consider that in its verification proof helps a mathematician convince others. This is an important aspect of a mathematicians' work because the testing of conjectures made after pattern identification helps validate the cases where the claims hold. Within the reasoning and proof processes there exists some uncertainty when one makes a claim (Stylianides, 2009). As a result the testing of claims is important to help validate observed patterns and see the limitations or possible extension of ones claims to all cases or mathematical situations. Students' view that in verifying a statement; a mathematician fulfills the obligation of establishing the validity of their claims.

Appreciation and Intellectual Necessity of Proof

The student's conception of the role of appreciation of mathematics demonstrated their understanding of the intellectual necessity of the mathematical knowledge they acquire at school. Harel and Tall (1991) identified the intellectual necessity principle as a standard for pedagogy that involves presenting subject matter in a way that encourages learners to see its intellectual requisite in their mathematical experiences. Students seemed to understand the role of proof based on their teachers' construction of proof arguments. According to several researchers, the status of proof will be enhanced in classrooms if it has personal meaning, explanatory power,

and value in students' mathematical experiences (e.g., Bell, 1976; Knuth, 2002; Schoenfeld, 1994). In light of these claims, I argue that students' talk about the role of proof in their appreciation of mathematics highlights their understanding of the intellectual need for proof in learning of mathematics. Students claimed that in this role they value (a) why a result is needed in mathematics, (b) how to effectively use a proven result in problem solving, and (c) the necessary connections between other proven results within the axiomatic system of results. This is important because several studies show that even though students develop competencies to write proofs, they may not appreciate why proofs are needed in their learning of mathematics (e.g., Fujita & Jones, 2014; Healy & Hoyles, 2009; Otten et al., 2014). Therefore students' identification of this role provides evidence that students mathematical experiences could possibly help them to value the necessity of proof in their learning of mathematics.

Generality of Proof

Within students' discussions about their personal proof choice, 16 of the 21 students selected non-general arguments. In particular, students opted to select argument four mainly because of the level of difficulty. By this student referred to the arguments use of algebraic operations with known angles in a specific case of a 30-60-90. Although this argument had the limitation of only addressing a specific case, students felt the mathematics knowledge used by the author best suited their mathematical ability. This was interesting in that all 14 of the 16 students were from the lower level classes in my study. This suggests that students did not acquire the necessary mathematical logical skills and knowledge that could support their understanding of the more difficult arguments. Nonetheless, the students did indicate that this argument used previously proven results, included all necessary steps, and logically supported the conclusion.

Despite the limitations of argument four, students did address the limitations of argument one. In particular, students stated that argument used measuring of the angles in a few generated triangles. Students indicated, “measuring is not proving” according to one student Safiyah. González and Herbst (2009) noted that Geometry typically does not allow students to use measurements. However measurements can provide an entry point for students to engage in proving. Students should base their conclusions on theoretical considerations rather than measurements. Measurements as in the case of argument 1 can prevent students from developing geometrical properties that can be generalized to an entire set of geometrical objects. Chazan and Houde (1989) also cautioned teachers to not use measurements as a means of developing statements within an argument in a proof. Chazan and Houde made a distinction between measuring and proving. These researchers argued that measuring does not provide evidence to support a claim in a proof. If students develop the inclination that measuring is proving they may rely on measurements rather than using geometrical theorems and properties to support their claims in a proof. Several students in this study, such as Safiyah, Faye and Jordan indicated their understanding that these two activities are not the same. Safiyah who selected a non-general argument (i.e., argument four), rejected argument one based on her belief that “measuring is not proving” On the other hand, Faye and Jordan selected general arguments as their personal proof choice. These students stated that argument 1 was not a valid option because it used measurements and this does not suffice as a proof of the theorem. Therefore despite students’ selection of the other non-general argument or general arguments as their personal choice, they all seemed to understand the limitation of argument one in the authors’ use of measurements to prove the theorem

Teachers' influence on students' criterion for validity

In this study, students were given four options of proof arguments to determine their best mark proof choice. The findings of these suggested in a similar manner to Healy and Hoyles (2000) and McCrone and Martin (2009), students selected the more formal looking deductive arguments. Students in this study highlighted several reasons for their selections. These included, (a) the generality, (b) inclusion of all statements, (c) inherent logic, (d) the use of previously proven results, and (e) the level of difficulty as reasons for their selection. The aforementioned reasons demonstrate students' perceptions of their teachers' criteria for a valid proof argument. In this regard, students seemed to consider the generality as a main factor in their selection. It is interesting that students based their selection of general arguments as their personal choice on their teachers' previous assessment of their proof writing skills. This suggests the influence of the teachers' expectations on students' development of their own proof writing abilities.

Additionally, when students' indicated their teachers' influence on their selections of either their personal or best mark proof choice they demonstrated their adherence to the meta-discursive rules of the discipline. By meta-discursive rules, I refer to Sfard's (2000) definition of the guidelines that govern the exchange of mathematical knowledge within a mathematical community. Sfard noted that differences in classroom environments, teachers' expectations, and curriculum demands can lead to differences in students' perceptions of the meta-discursive rules. With regard to the case of reasoning and proof, I consider how these meta-discursive rules influence students understanding of structure, validity, and generality of a proof. For example, a teacher or another authority such as a textbook or examiners may set guidelines that constitute what counts as a valid proof. In the case of Trinidad and Tobago, it seemed that teachers and

external authorities such as CSEC examiners stipulate what counts as an acceptable proof. Given the high stakes of the CSEC exams it seems beneficial for students to consider and adhere to these guidelines when producing proof arguments.

Although only 5 students showed agreement with the personal and best mark proof choice, their selections seemed to be influenced by the meta-discursive rules of the classrooms in which they participate. The students' selections provided some evidence of their learning with proof. For example, McCrone and Martin (2009) hypothesized that if the learning objectives of a classroom are successfully achieved then student's communication about proof should reflect the teachers' expectations and meta-discursive rules. However if there is disagreement with the meta-discursive rules of the classroom in Trinidad and Tobago, then researchers may gain some insight into possible reasons for students' low performance of proof-based items in their school based assessments and possibly the CSEC examinations.

Of the 21 students, three students' personal proof choice was in agreement with the best mark proof choice. Two students selected the argument two based on its' generality and use of previously proven results. Both students stated that the aforementioned characteristics would be appealing to their teacher's grading criteria. Whereas one student selected argument four, she seemed unaware of the limitation of its' generality. Melissa's selection was based on her adherence to her teacher's recommendation to include all steps when writing a proof. As a result, Melissa considered argument four as meeting her teacher's expectation and deserving of the highest mark. Martin et al., (2005) suggested that in a proof-based classroom, a teacher's responsibility is to help their students develop habitual inclinations that produce mathematically valid arguments that meet the teachers' criterion for validity. However, teachers need to set expectations that are feasible and conductive to their students' ability. Students as those in this

study must be accountable for developing an understanding of the process of proof constructions and the requirements of valid proofs. This understanding will be beneficial to them producing proofs that are valid and meet their teachers' or better yet the external examiners' criteria. My analysis of students' conceptions suggests that students seem to be aware of their teachers' expectations and thus adhere to these in making their personal proof choice. Because students are faced with learning proof processes and understanding their teachers' and examiners expectations, it is often difficult to discern whether the difficulties students face with proofs are due to the struggles with (a) the Geometry content, (b) understanding the formal definitions, theorems or logical links, or (d) disparities between their considerations of what counts as a proof and their teachers' or examiners criterion of validity of a proof. All the aforementioned struggles coupled with the reformers' vision for proof to be an integral part of students mathematical experiences could create a daunting situation for educators and learners. If students are unable to produce proofs that meet their teachers expectations as well as master the processes of proof construction they will not see the intellectual value of proof in their mathematical learning. As a result the reformers' vision for students' learning for proof may not be met.

Conclusion

In this study, my overarching question was "What is the nature of secondary school students' conceptions of proof in Trinidad and Tobago?" To answer this question I considered the issues embodied within the implications of the recent mathematics education reform on student's learning of proof. Figure 4.9 shows the overarching issues I considered in my inquiry into students' conceptions. The three issues that could possibly influence the students' conceptions were (a) reform policies on reasoning and proof, (b) students' difficulties with proof, and (c) CSEC's opportunities for proof. My analysis suggests that aforementioned issues

influence students learning. The reformers of mathematics education in Trinidad and Tobago suggest that students should have more opportunities with proof in their secondary school mathematical experiences of proof in Trinidad and Tobago (Republic of Trinidad and Tobago, Ministry of Education, 2003, 2009). Despite the reformers' call for proof to play a fundamental role in school mathematics, some students have difficulties in appreciating the importance of proof in their mathematical experiences and understanding the role of proof. This is evident in students opting to not answer proof-based items in the Caribbean Secondary Examination Certificate (CSEC) exams and the low performance of those who attempt these items (CXC, 2013, 2014).

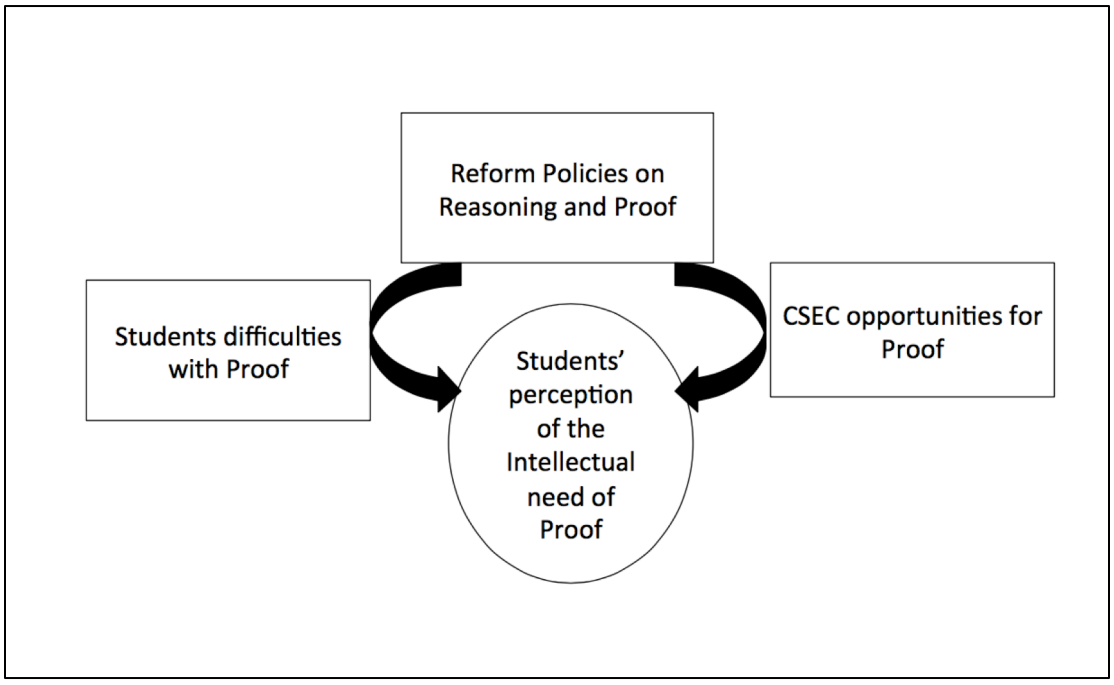


Figure 4.9: Overarching issues surrounding students' conceptions of proof.

In this study I argue that according to the reformers' recommendations, and students performance in CSEC, there exists a need for students to understand the intellectual necessity of proof in their mathematical experiences. In particular within an educational system that is centralized, it is important that the case of Trinidad and Tobago, the homogeneity of the

curriculum materials presents a unique perspective of how opportunities for proof in the curriculum and CSEC exams' influence on students' conceptions of proof. The findings of this study demonstrate that students' conceptions of the roles of proof suggest their perception of the intellectual need of proof in school mathematics. Students seem to consider proof as the link between their previous and current knowledge, which is evident in their teacher's efficient application of the knowledge while constructing proofs. Students seem to see proof beyond arbitrary exercises given by their teacher. Students also seem to understand that proof is integral to their understanding of mathematical relationships and their metacognitive development of proof writing skills. Within the optional geometry section of the curriculum, students are required to engage in the all-encompassing activities of reasoning and proof. Although the policy documents promote proof as integral, the recommendations present proof as the high point of engagement in reasoning and proof processes. This is evident in the recent prevalence of questions that promote the development of rationales, which explain students' mathematical thinking as they use algebraic operations and geometry theorems or definitions to solve problems. Students in this study recognized that these informal explanations present possible opportunities for proof construction. Therefore students see the intellectual need for explaining one's reasoning processes and validating one's claims.

As suggested by CXC examiners, teachers should provide extensive opportunities for students to explain their reasoning and use their geometrical knowledge to support their claims (CXC, 2014). This recommendation seems to advocate the need for students to develop habits of mind that always motivate them to explain why their claims are valid. As students position themselves as authors of proof, they seemed to highlight the necessity to always explain their thinking so that (a) other readers can understand their line of reasoning, (b) fulfill the

expectations of their teachers regarding good proof writing habits, (c) appeal to the examiner's criteria for valid answers to exam questions, and (d) mirror the work of real mathematicians. However, we are yet to fully understand why students continue to perform poorly on CSEC examination questions or opt not to answer these questions. Nonetheless this situation begs further investigation with a larger sample of students and possibly the direct links between instruction, the demands of the CSEC examination, and students' performance. In this study, I gathered evidence of students' learning through their conceptions about proof. In a future study I can use a larger sample and directly measure students learning with proof. In this case, we may find direct explanations for students' performance with proof. Moreover teaching and curriculum materials impact students' learning; therefore it is worthwhile considering all these components, as we understand students' conceptions of proof in Trinidad and Tobago and by extension any similar centralized education system.

CHAPTER 5:

CONCLUSION

In this dissertation my overarching question asked: What are the implications of reform on the teaching and learning of secondary school mathematics in Trinidad and Tobago? I answered this question by considering the case of reasoning and proof to investigate the curriculum, teaching, and students' learning in three interrelated studies. Figure 5.1 shows a diagram of the three studies that contributed to answering my overarching research question. First, I examined the nature of the opportunities for reasoning and proof embedded in the three recommended secondary school textbooks in Trinidad and Tobago. I adapted an analytic framework established by Otten, Gilbertson, Males, and Clark (2014) to investigate the reasoning and proof opportunities in the Geometry sections of the textbooks. Second, I considered theories on sociomathematical norms (Cobb et al., 199), teachers' actions during the instruction of proof (Martin & McCrone, 2003, 2005), and characteristics of reform-based teaching (Hufferd-Ackles, Fuson, & Sherin, 2004) to examine the nature of the teaching of reasoning and proof in secondary school. I constructed and analyzed descriptive cases studies of three teachers' instruction of the topic Congruency of Triangles. My analysis focused on (a) the classroom microculture (i.e., classroom mathematical practices and sociomathematical norms), (b) the teachers' pedagogical decisions, and (c) their use of CSEC examination materials and textbooks for creating opportunities for reasoning and proof. I also focused on any existing characteristics of reform-based mathematics teaching as defined by Hufferd-Ackles et al. (2004). In the third study, I conducted semi-structured interviews with 21 students. I gathered evidence of students' conceptions of, (a) the roles of proof in school mathematics, (b) the structure,

validity, and generality of proof, and (c) the opportunities for reasoning and proof in the CSEC examinations and textbooks.

Several researchers claim that despite the complex interaction among the content of mathematics textbooks, teachers, and students, there exists an inextricable linkage between the content of textbooks and student learning (Moyer, Cai, Wang, & Nie, 2011; Schmidt et al., 2001; Stein, Remillard, & Smith, 2007). This dissertation adds to this work by examining the relationship between the textbooks, teaching, and student learning with regard to the case of reasoning and proof. This dissertation also adds to the ongoing dialogue about the role of curriculum materials in teachers' agency of the reformers' policies and the possible influences on students' opportunities to learn reasoning and proof. Although, it may seem that the warrants of this dissertation are specific to Trinidad and Tobago, I argue that it is a case of the bigger issues surrounding the interplay of policy and practice within an educational system of unique characteristics. These characteristics include (a) the prescriptive instructional guidance afforded to teachers, (b) the centralized governing educational system, (c) the uniformity of the prescription of textbooks and teaching materials, and (d) the homogeneity of assessment materials. Moreover due to the non-existing studies investigating specifically the teaching of mathematics before and after the reform, my findings do suggest direct consequences of the reform. The limited availability of studies is unable to support comparisons of pre- and post-reform instruction, which may support conclusions about instruction resulting from reform-oriented policies. Therefore I posit that the evident teaching described in this dissertation, is not possibly due to the reform but more a presentation of examples of teachers' adoption of instructional policies aligned with the reformers' vision. In the following sections I discuss overall findings from the three studies. I discuss implications of these findings and possibilities

for extending this work. These include the implications of (a) the GCNE exercises in the recommended textbooks and exams, (b) current teaching practices and the role of future professional development for teachers, and (c) students' conceptions and their learning of mathematics.

Findings on Curriculum Materials

My analysis of the three textbooks suggests that there exists opportunities for students to engage in or reflect on reasoning and proof. However, the type of opportunities varied across the three textbooks. For example, the more recent published textbooks seem to exemplify more opportunities for reasoning and proof. The Mathematics a Complete Course (MCC) and Mathematics for CSEC (MCSEC) both exemplify various processes of reasoning and proof. These included the authors' offering activities that allow students to engage in pattern identification, conjecturing, and developing rationales (i.e., non-proof arguments). This finding is important because it suggests that these textbooks' authors provide the types of scaffolding students need to eventually construct a proof argument. By finding patterns, students can formulate new conjectures and test these conjectures against new evidence, thus leading to the revision or validation of these conjectures. The validation may initially take the form of developing rationales that could develop into the construction of a proof. Several researchers advocate that these activities are important for building the foundations for students' metacognitive development of writing proofs (e.g., Bieda, 2010; Chazan, 1993; Cirillo & Herbst, 2008; Stylianides, 2009).

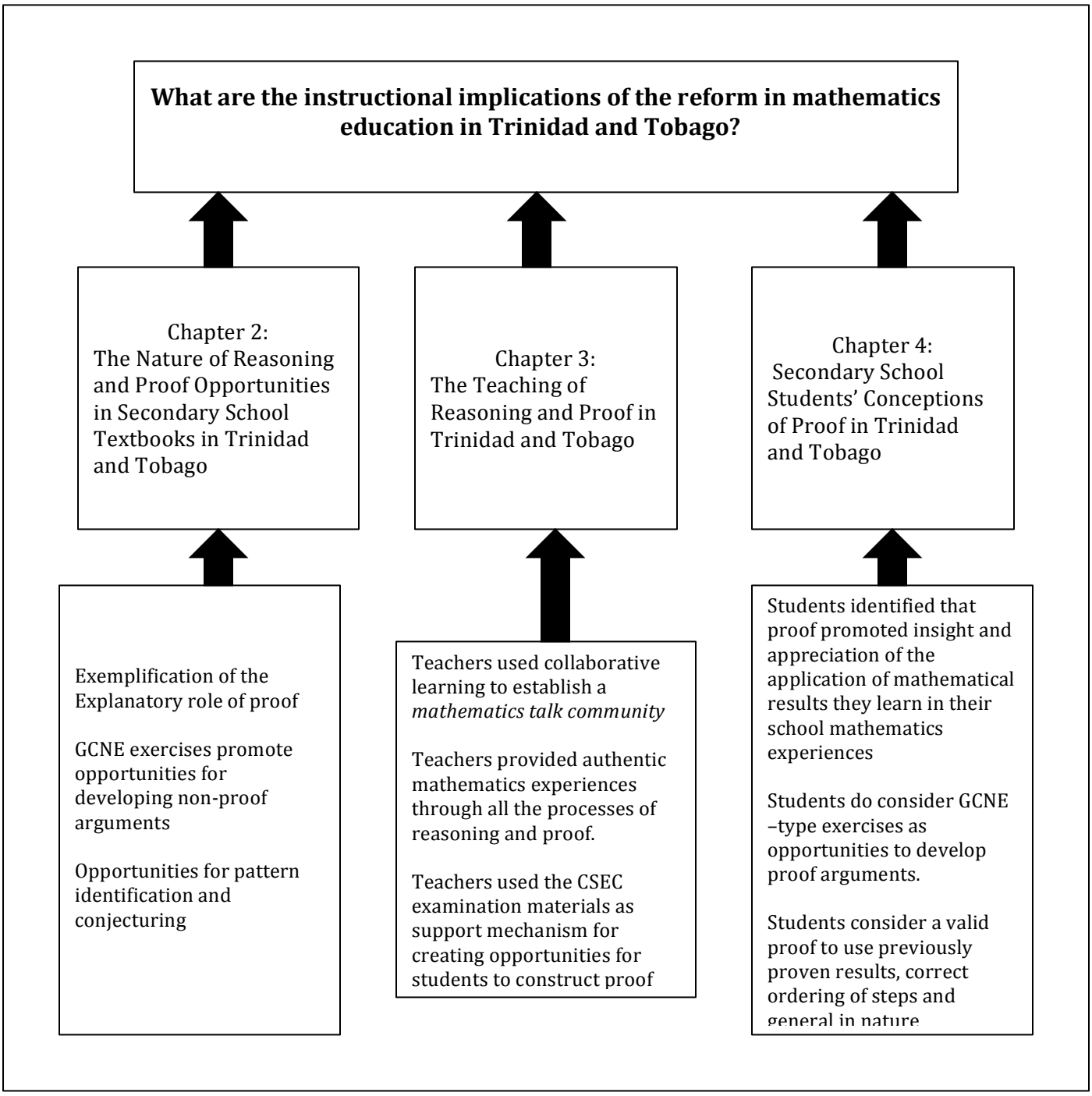


Figure 5:1 Overarching map of studies addressing the implications of reform of mathematics education in Trinidad and Tobago.

Additionally, the textbooks' presentation of proof in conjunction with these activities of pattern identification, conjecturing, and rationale development, could help students see the

intellectual need of proof in a mathematicians work. Several researchers commend the aforementioned characteristics of textbooks, which offer the all-encompassing activities of reasoning and proof (Bieda, 2010; Otten et al., 2014; Sears & Chávez, 2014; Stylianides, 2009). These researchers agree that when textbook authors, provide opportunities for students' engagement in the trajectory of processes, students gain experience in moving from inductive to deductive forms of reasoning in developing mathematical knowledge.

Both MCC and MSEC contained opportunities for construction of proof however it was not as substantial as the earlier published textbook. The Certificate Mathematics (CM) offered the most opportunities for proof construction among the three textbooks. The authors' emphasis on the construction of proof aligns with the reform recommendations, "students must be given opportunities to develop logical deductive arguments" (Republic of Trinidad and Tobago, 2009, § 2: 1). However in the same vein the reformers also stated, "students should have extensive opportunities to identify patterns, make, and test conjectures, critique arguments and provide reasoned explanations and construct proofs" (Republic of Trinidad and Tobago, 2009, § 2: 1). The latter quote seems to suggest that although students should engage in constructing proofs, they should have opportunities to engage in the all-encompassing activities of reasoning and proof.

In light of these recommendations and the existing opportunities for reasoning and proof, my findings suggest that the inclusion of all three textbooks as the prescribed materials provide support for teachers meeting the demands of the reform policies. The larger issue is that teachers may use these textbooks differently and may not be aware of the strengths and weakness of the varying textbooks. Furthermore there are neither teacher's editions of the recommended textbooks nor commentaries in the student's editions to support teachers' instruction when using

the textbooks. The findings of this study may help in this regard because policy makers, researchers, and educational stakeholders will understand the quality and quantity of the existing opportunities for reasoning and proof in the textbooks.

For example, in the future selection of textbooks for the Textbook Rental Program (TRP) , the findings about the quality and quantity of opportunities for reasoning and proof could help in understanding the strengths and weaknesses of the existing textbooks. This understanding can lead to possible revisions or development of textbooks that support the intended reformers' vision for the teaching and learning of reasoning and proof. Furthermore the findings of this study have the potential to guide the Ministry of Educations' recommendations of textbooks for the TRP. Due to the prescriptive guidance afforded to teachers' through the selection of textbooks in the TRP, it is important that MOE stake holders are informed about how the textbooks' align with the reformers' recommendations for reasoning and proof in secondary school mathematics.

Furthermore these findings can help textbook authors in their future textbook development and creation of policies addressing the teaching and learning of proof in Trinidad and Tobago. For example, by understanding the possible deficiencies of the existing textbooks in their promotion of reasoning and proof, textbook authors could develop future textbooks that address the deficiencies while supporting the reformers' recommendations for reasoning and proof in secondary school textbooks. Additionally the current textbooks' authors could possibly use the findings of this study to guide their future revisions of their textbooks. Overall the findings support the future development of textbooks that support the teaching and learning of reasoning and proof.

Geometric Calculation with Number and Explanation (GCNE)

Another important finding of this study was the discovery of Geometric Calculation with Number and Explanation (GCNE) exercises. As I described earlier in the dissertation, GCNE exercises were prevalent in the MCC and MCSEC textbooks. The GCNE or simply calculate and explain exercises afford the development of reasoned explanations which justify the calculations of unknown measurements in geometrical figures. The GCNE exercises have the potential expectation of students using geometrical concepts, theorems, and definitions to formulate arguments explaining why a resulting computation is valid.

My analysis of the textbooks suggests that GCNE tasks are frequently offered as opportunities for students to engage in developing rationales or reasoned explanations. This finding is important because the GCNE task have the potential to develop from less formal explanations to a proof. Therefore the opportunities for solving GCNE tasks have the potential to become opportunities for students to construct proofs. Additionally GCNE tasks exemplify the explanatory role of proof in mathematics. Several researchers claim the status of proof will be elevated if most and foremost its explanatory role is promoted in school mathematics (e.g., Bell, 1976; Hanna, 1990; Hersh, 1993). Therefore the exemplification of explanation helps students understand why a result is valid and promotes insight into the relevance and use of geometrical concepts when solving problems.

A major feature of the GCNE tasks found in the textbooks was an accompanying diagram, which can initiate physical and mental processes that lead to deductive reasoning about its geometrical properties. The inclusion of diagrams in GCNE tasks constitutes an important dimension of *cognitive complexity* that requires high-level thought and reasoning of students (Magone, Cai, Silver, & Wang, 1994). As I explained in Chapter 2, cognitive complexity refers

to the features of a mathematical task that allows students engagement in cognitive process such as making connections among geometrical concepts and mathematical reasoning. Therefore, solving GCNE tasks has the potential for students to reason with and about relationships between the givens and the unknown features in a geometrical diagram. The inherent problem-solving process of GCNE tasks provides students with the opportunity to use algebraic operations with connections to geometric theorems and concept. Additionally, students can use multiple-step reasoning to justify the steps in their computations. Several researchers claim that the aforementioned properties of solving GCNE tasks are characteristics of highly complex cognitive demand (Henningsen & Stein, 1997; Hsu, 2014). Therefore, the prevalence of GCNE tasks has the potential for students developing arguments that could eventually be considered a proof and students' engagement in highly complex cognitive activity. However, these characteristics lead to the question of whether students realize that such informal tasks could lead to their development of proofs although they do not formally ask students "to prove." Additionally it is worth investigating whether teachers see the potential of these GCNE tasks in helping their students' reasoning and proof skills.

Despite the potential of the GCNE tasks for engaging students in constructing proofs, we are yet to fully understand why students continue to perform poorly on CSEC examination proof items. Therefore there exists the need for further evaluation of GCNE opportunities in their affordance of proof construction. Additionally there exists the need to further examine the CSEC examination and determine whether there may be agreement or disagreement of the proof content in the textbooks, curriculum, and examiners' expectations of students. By carefully examining these materials, researchers can possibly find explanations for the poor students' performance on proof items.

Implications and Future Work about Analysis of Curriculum Materials

Overall my findings about the opportunities for reasoning and proof in the textbooks can potentially help in guiding the design of future textbooks and instructional resources, which offer opportunities for students to see the intellectual need of reasoning and proof in their mathematical experiences. My future work can look at the opportunities for reasoning and proof in other content areas such as Algebra.

In the secondary school mathematics curriculum, Algebra is considered another area outside of Geometry, whereby students can engage in reasoning and proof (Chazan, 2000; Kieran, 1992). Chazan (2000) discussed that by placing emphasis on reasoning and proof in areas outside of Geometry such as Algebra, students may appreciated the intellectually necessity of these processes in their learning of secondary school mathematics. Kieran (1992) concurs with Chazan by her arguments, which support the need for proof to take a pivotal role in developing students reasoning skills in other content areas such as Algebra. Kieran makes the case that the Algebra course in the US secondary mathematics curriculum can be tailored to help students construct proof arguments and engage in scaffolding activities of pattern identification and conjecturing.

In the case of Trinidad and Tobago, the reformers also make a case for the integral role of reasoning and proof in Algebra. According to the reformers, students need to have opportunities to "make and investigate conjectures, develop and evaluate mathematical arguments and proof, and reason about mathematical ideas in Number Theory, Algebra and Geometry" (Republic of Trinidad and Tobago, Ministry of Education, 2009, § 2:8). Additionally the CXC examiners also consider Algebra as a place in the curriculum for students' engagement with reasoning and proof (CXC, 2012). For example, in the recent examiners' report on students' performance, there exist

questions, which ask students to identify patterns and write a conjecture or rule for the observed algebraic pattern (see Figure 5.2).

A number sequence may be formed by counting the number of dots used to draw each of a set of geometric figures. The first three figures are shown below.

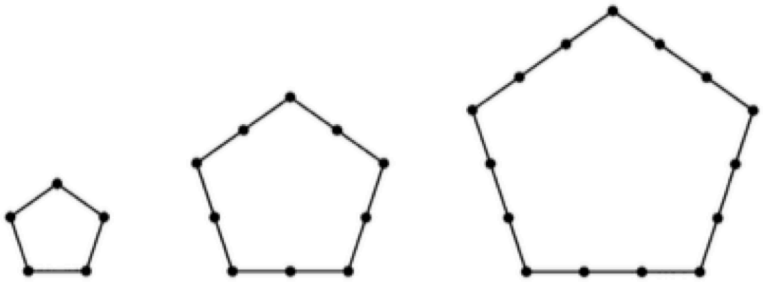


Figure 1 **Figure 2** **Figure 3**

On the answer sheet provided,

(a) Draw Figure 4, the next figure in the sequence above. **(2 marks)**

(b) Complete the table below by inserting the missing information at the rows numbered (i) and (ii).

	Figure (*f*)	**Total Number of Dots**	
		Formula	**Number (*n*)**
	1	$5 \times 2 - 5$	5
	2	$5 \times 3 - 5$	10
	3	$5 \times 4 - 5$	15
	4	Not Required	Not Required
(i)	5		
(ii)	6		

(4 marks)

(c) Write an expression in *f* for the number (*n*) of dots used in drawing the *f*th figure. **(2 marks)**

(d) Which figure in the sequence contains 145 dots? **(2 marks)**

Total 10 marks

Figure 5.2: Example of a CSEC reasoning and proof question adapted from CSEC June 2014.

In this question students are provided with a train of geometrical shapes and are asked to write a generalized algebraic formula for the *fth* geometrical object in the sequence. Although the question presents geometrical shapes, the examiners seem to motivate students to use algebraic knowledge to formulate the generalized formula or conjecture. According to the examiners' report for this examination question, "Teachers need to provide opportunities for students to continually develop reasoning skills using algebraic operations for pattern identification and conjecturing " (p. 10, CXC, 2014). This quote suggests that Algebra can provide a place in the curriculum for students to engage in and reflect on reasoning and proof.

As a result of the aforementioned arguments, I posit that it would be useful in my future work to make useful comparisons between the affordances in Geometry and Algebra in the respective textbooks and national curriculum. Furthermore it may be useful to analyze the CSEC examination questions to determine the type of opportunities for reasoning and proof. Despite the limitations of the textbooks applicable to only Trinidad and Tobago, my current and future work could inform other educational domains with a similar education system. These educational domains would consist of a centralized curriculum and examinations along with a central governing body's prescriptive guidance for teaching and learning.

Establishing Mathematics Talk Communities

My analysis of three teachers' instruction of reasoning and proof indicate that teachers used collaborative learning to establish a mathematics talk community (Hufferd-Ackles, Fuson, & Sherin, 2004). Within such a community teachers established norms that facilitated students becoming active questioners of each other. Teachers established social norms that relied the expectation that all ideas must have accompanying explanations that support one's reasoning processes. Teachers also considered students as sources of mathematical ideas and teachers gave

students the autonomy to critique, develop, and enhance each other's mathematical ideas. The aforementioned characteristics align with Hufferd-Ackles and colleagues' descriptions of reform-based teaching of mathematics.

The reformers in Trinidad and Tobago recommend that students should have the responsibility of actively participating in the construction of mathematical knowledge. Students must possess a view of mathematics that supports connections with mathematical concepts and experiences outside of mathematics (Republic of Trinidad and Tobago, Ministry of Education, 2003). The reformers' view of learning suggests that teachers are no longer required to present content through direct instruction or demonstrations, but must create the conditions, which allow students to take their own mathematical actions to construct knowledge. In such instructional situations, teachers must play more the role of classroom facilitator and guide (Ball & Chazan, 1994; Borko et al., 1992; Schifter, 1998; Smith, 1996). To address these recommendations, teachers in my study seemed to create a social constructivist environment to learning. This resulted in teachers' use of group work to create opportunities for collaborative learning. Teachers also afforded students the opportunities to be the expert or more knowledgeable other among smaller groups of learners and in come cases for the entire class. Students were held responsible for each other's learning and were considered as the source of mathematical ideas. All the aforementioned actions of both teacher and students suggest that there are elements of reform-based instruction in Trinidad and Tobago.

Authentic Mathematics Experiences

My analysis of teachers' instruction in Chapter 3 suggests that the teachers create opportunities for authentic mathematics experiences. By this I refer to the teachers' offering of opportunities to engage in the all-encompassing activities of reasoning and proof. In particular,

the teachers' emphasis on pattern identification and conjecturing allowed students to engage in activities similar to the type of work of real mathematicians. This was important for engaging students in scaffolding activities that motivate their development of proof. Additionally the authentic experiences support students' appreciation of the intellectual need of proof in the development of mathematical knowledge. As a result, proof goes beyond simply confirming the validity of already proven results but as a methodological necessity for validating one's claims and mathematical hypotheses. The three teachers in the study, emphasized pattern identification and conjecturing during their instruction. In some cases the teacher directly used the conjecturing activities in the textbooks. For example, one teacher used a suggested activity in Toolsie (2009) for making conjectures about the sum of interior angles in a triangle. This activity allowed students to identify patterns, make, and test conjectures before writing a reasoned explanation supporting the validity of their conjectures. As a result, students had an authentic mathematics opportunity to establish the geometric theorem about the sum of interior angles. Therefore, it seems that the exemplification of the scaffolding activities in the textbook influenced and supported the teacher's creation of authentic mathematics experiences. Cirillo (2008) noted that in order for teachers to effectively create authentic mathematics experiences, they need the type of instructional support that fosters students' engagement in all the processes of reasoning and proof. Therefore I posit that the textbooks authors' exemplification of pattern identification, conjecturing, and eventually proof construction, provides the necessary support for teachers' creation of authentic mathematics experiences in Geometry.

The Interaction between Curriculum Materials and Teaching

The uniformity of the prescription of textbooks and teaching materials and homogeneity of access to these materials seem to suggest that these curriculum materials are indeed potent

agents of reform policy in Trinidad and Tobago. Therefore, Trinidad and Tobago provides a good example of the role of curriculum materials in helping teachers meet the demands of reform. As I explained earlier, Trinidad and Tobago's, educational system offers very prescriptive guidance for content coverage (Jennings 1993; Lewin, Keller, & Taylor, 2002). The centralized governing body The Ministry of Education (MOE) offers this guidance through the national curriculum. This document provides extensive and focused guidance about instructional content. Additionally the MOE in collaboration with the external examiners, The Caribbean Examination Council (CXC) prescribes the textbooks and mathematics syllabus for the Caribbean Secondary Education Certificate (CSEC) examinations. Furthermore, the MOE ensures homogeneity of access to these support materials through the Textbook Rental Program (TRP). Through this government initiative all secondary schools are provided with the recommended textbooks, CXC support materials such as the mathematics syllabus, examiners reports, and CSEC examination preparatory materials (Republic of Trinidad and Tobago, Ministry of Education, 2003, 2009). Each school has access to these resources. As a result, the dissemination of the reform-oriented policies seems homogeneous. In this case, textbooks, CSEC examination materials, and curriculum guides offer extensive and focused guidance about instructional content.

Past studies indicate that in countries with a similar educational context as Trinidad and Tobago, teachers adhere to the prescriptive guidance; thus content coverage is relatively homogeneous (Lubienski, 2011; Otten, Gilbertson, Males, & Clark 2014; Porter, Floden, Freeman, Schmidt, & Schwille, 1988; Schwille et al., 1983). My findings of Chapter 3 of this dissertation suggest that teachers attend to these materials, however their use of the materials for actually teaching varied. In some cases teachers used the textbooks directly for assigning

homework and in class activities, whereas in other cases the teachers used the textbooks only as resource for preparing their lessons while opting to create worksheets from varied resources such an online support and the CSEC examination preparatory materials.

My findings about teacher's use of curriculum materials coupled with my analysis of the textbooks from Chapter 2 suggest that teachers use the existing opportunities as resource materials for assigning in class activities and homework. However, teachers seem to use the CSEC examination materials to create the necessary opportunities for constructing proof arguments. Past studies demonstrate that the emphasis placed on reasoning and proof differs across geometry content of textbooks in countries around the world (Fujita & Jones, 2014; Hanna & de Bruyn, 1999; Otten et al., 2014). The way reasoning and proof is treated is influenced by the context of the classroom learning environment, students' ability, and demands of the curriculum (Bieda, 2010; Jones et al., 2006; Stylianides, 2009). Researchers noted from their examination of the implementation of reform recommendations in classrooms, that students are seldom afforded the opportunities to engage with meaningful experiences with reasoning and proof (Bieda, 2010; Heinze & Ross, 2004). Therefore reform-oriented interventions mediated through textbooks may or may not be sufficient to be a catalyst for instructional change and students' increased engagement with reasoning and proof (Bieda, 2010; Remillard, 2000; Sears & Chávez, 2014;).

Based on my findings, it seems that the curriculum materials do play an important role in teachers' creation of opportunities for reasoning and proof. Therefore the prevalence of GCNE opportunities could support teachers in their expectations of students to always explain their mathematical ideas. As stated in the CXC examiner's report, "teachers should provide ample opportunities for students to practice stating the theoretical reasons for calculations derived from

the geometry of plane figures" (CXC, 2014, p.11). This suggests that given the need for students to calculate and explain in Geometry, and teachers' adherence to the examiners' recommendations, the GCNE exercises afforded in the textbooks provide the support necessary for teachers' instruction and exemplification of the explanatory role of proof.

Implications and Future Work on Teaching

As I stated earlier in this dissertation, several researchers hypothesized that a centralized governing body with a national curriculum and assessments may provide the solution to the plethora of issues of reform (Cohen, 2011, Cohen & Hill, 2000; Lubienski, 2011). In particular, Cohen (2011) argues that teachers in the United States have difficulty meeting the reformers' recommendations due to the fragmented nature of school governance coupled with the lack of homogeneity of instructional support and assessments. Lubienski's (2011) argument that that even in such situations, as in Trinidad and Tobago, teachers may have varying interpretations and use of the prescribed instructional support. However, my findings suggests that the three teachers' interpretations of the prescribed instructional materials coupled with their expertise seemed to demonstrate instruction that aligns with the reformers' vision for reasoning and proof. The teachers' use of the curriculum seems to promote opportunities that are beneficial to students' development of reasoning and proof skills. My current work could not provide sufficient evidence to fully substantiate how the curriculum materials impacted the planning and implementation of opportunities for reasoning and proof However as a future project, it would be worthwhile examining the extent of teachers' use of the prescriptive guidance and instructional support. This proposed future study would look at policy implementation and enactment and could provide insight into the dynamics of the written curriculum, teachers' agency and the enacted curriculum.

Researchers suggest that teachers pedagogical decisions about what to implement in their classrooms is mediated through the curriculum materials that are available (Beaton et al., 1996; Lubienski, 2011; Remillard, 2000; Stein, Grover, & Henningsen, 1996). Therefore the existing opportunities for reasoning and proof may provide the support needed to help teachers' agency of the reformers' vision for students' learning with reasoning and proof in Trinidad and Tobago. However there exists the need to further investigate the various ways teachers use these materials to support their agency of the reformers' ideals. My findings suggest that teachers also use various external resources such as NCTM's Illuminations website. Given that the textbooks I examined in Chapter 2, do not have accompanying instructional guides and solution manuals, it would be useful to understand the influence of external resources on teachers' creation of opportunities for reasoning and proof and their agency of the reformers' instructional policies. Teachers' use of the CSEC materials suggests a major role in influencing students' engagement with proof. However, this is an area that also needs further investigation to determine exactly how the CSEC materials influences practice and the extent of the possible influences on instruction and student learning of all areas of mathematics.

Darling and Hammond (2008) noted that reform could not succeed unless it focuses on creating the school conditions in which teachers can teach effectively. Experienced teachers may use their own models of effective teaching to interpret the reformers' ideals and incorporate those elements consistent with their views (Prawat, 1992; Remillard, 2000). The act of teaching to reform is a complex interaction of numerous elements such as the teachers' content and pedagogical knowledge and how they fit into the instructional policies of the reform. I posit that it would be beneficial in a future study to examine the role of teachers' content and pedagogical knowledge in their agency of reform recommendations in Trinidad and Tobago.

A Case for Future Professional Development

As I described earlier, the teachers in this study demonstrated teaching aligned with the reformers' vision. Each teacher seemed to possess expertise that was well-respected and valued among his or her fellow teachers at their respective school. As a result, I posit that given the evidence that support the teachers' exemplification of desirable teaching practices that advocate the reformers' objectives for the teaching and learning of proof, there exists a need for these teachers to share their expertise with other teachers.

Professional development programs within schools and across schools can highlight these teachers' *practitioner knowledge* (Hiebert, Gallimore & Stigler, 2002). By teachers' practitioner knowledge, I refer to the skills teachers develop as they solve problems related to the teaching of specific content. Although this type of knowledge is specific to the classroom environment, students' ability, and content being taught, it can be translated to a professional knowledge base that can be useful and applicable and transferable in one context or classroom into another. By a professional knowledge base I refer to the documentation of teachers' expertise and understanding of effective pedagogical practices that aid in achieving learning objectives and implementing instructional policies (Hiebert et al., 2002). In mathematics teaching, teachers' practitioner knowledge can be essential to understanding the dynamics of students mathematical performance, teaching practices, and meeting the demands of reform (Da Ponte & Chapman, 2006). This knowledge can be publicly shared and made accessible to all teachers as professional knowledge so that they can use practitioner-based expertise to deal with problems associated with specific content areas. The teachers in this study possessed practitioner-based knowledge that can be transitioned to professional knowledge that can help other teachers' instruction of reasoning and proof in Trinidad and Tobago. Hiebert et al. (2002) makes the case that

professional development can be a means of facilitating the transitioning of teachers' practitioner knowledge to professional knowledge.

Based on the findings of this study, each teacher demonstrated unique pedagogical strategies and expertise that seemed to demonstrate the reformers' objectives for the teaching and learning of reasoning and proof. For example, the teachers used technological innovations, group activities, and hands-on interactive tasks to help create learning environments that align with the reformers' vision for the teaching of reasoning and proof. Given the existing practices that are associated with a common curriculum, I argue that through professional development opportunities, the teachers' expertise and pedagogical strategies can be represented in more generalizable forms that are applicable in classrooms across Trinidad and Tobago. I propose that through professional development opportunities within and across schools, teachers can collaborate to assist each other in solving problems of teaching reasoning and proof and even other content areas. As teachers mentor novice teachers, they can discuss practices that are most useful or essential to achieving learning objectives and learn about teaching practices aligned with the reformers' vision. Teachers' collaboration and mentoring can provide the settings in which teachers could learn professional knowledge that can be translated within and across schools. Therefore, I advocate that in Trinidad and Tobago professional development can provide the space for teachers to collaborate with other teachers and promote a transition from practitioner to professional knowledge.

With regard to reform and mathematics instruction, past studies present professional development as a means of enhancing or fixing teaching practices to meet the reformers' vision for instruction (e.g., Ding & Jones, 2006; Hoyles, 1997; Jones, Fujita, & Ding, 2006). However, I argue differently that professional development can provide an opportunity for teachers to share

their expertise and mentor fellow colleagues within and across schools in Trinidad and Tobago. As an alternative to research-based professional development, Hiebert et al. (2002) posited that opportunities for sharing professional knowledge affords the space for teachers to collaborate about instructional ideas, mentor each other, and document their successful pedagogical practices. Hiebert and colleagues noted that this alternative form of professional development involves the documentation of teachers' professional knowledge organized according to various content areas whereby other teachers can learn from other teachers' experiences and tested pedagogical practices. This organization is most suitable in an educational system with a centralized governance and national curriculum because these characteristics afford uniformity in content being taught and also a common framework by which teachers' can discuss and share their pedagogical practices and expertise. Furthermore a common framework affords the translation of professional knowledge from one classroom to another. Past studies have shown that in a centralized governed educational systems with a national curriculum such as Japan, professional development programs, for example Lesson Study[15] afforded opportunities for teachers to successfully transition their practitioner knowledge to professional knowledge (e.g., Fernandez & Yoshida, 2012; Lewis, 2002; Watanabe, 2002). Based on documented evidence, these professional development activities afforded the development of a professional knowledge base that helped teachers improve students' performance on a national and international scale (Hiebert et al., 2002).

In considering these ideas about the ideal educational situation for teacher-led professional development, I posit that the national curriculum of Trinidad and Tobago provides

[15] Lesson Study is a Japanese in-service professional development program that allowed elementary school teachers to meet in small groups to collaboratively plan, evaluate, implement and revise lesson surrounding various content areas in the national curriculum

an ideal situation whereby all teachers can critically examine instructional strategies that meet the goals of the curriculum and support students' learning. The homogeneity of the curriculum and instructional support available to all teachers provides the common framework whereby teachers can collaborate and share instructional ideas, lessons, and expertise that achieve common learning objectives and meet reform recommendations. This homogeneity can also allow teachers' documented "best practices" to be applicable in other teachers' classroom using the same curriculum and prescribed instructional support. I argue that organized teacher-led professional development activities at local schools and on a national scale can provide the space for teachers such as the participants in this study to share their practitioner knowledge.

Furthermore, there exists a next phase of the reform of the mathematics curriculum in Trinidad and Tobago. This includes the introduction of School-Based Assessment (SBA) projects in Mathematics CSEC syllabus. Examiners will assess students' performance on the examinations and submission of SBA projects. All schools would be required to implement the SBA component of the exam within the next two years. Students are expected to submit personalized projects using real-world experiences that demonstrate creative reasoning about mathematical content in Algebra, Geometry, and Number Theory. CXC examiners have provided sample exercises and project for instruction which afford students opportunities to develop skills in pattern identification, conjecturing, and developing reasoned arguments and proofs. Although the teachers in this dissertation demonstrated their ability to promote students engagement with reasoning and proof, there is the possibility that teachers in Trinidad and Tobago may need guidance in meeting the objectives of the SBA component. Some researchers in their examination of reform noted that when new components are introduced into the curriculum and assessments, teachers might need some form of instructional support and

guidance to help them meet the objectives of the new additions (e.g., Kennedy, 2005; Cohen, 2011). Given the future adoption of the new phase of the CSEC examination, there exists the possible need for professional development supporting teachers' preparation of students for the SBA component of the exam. I also advocate that in this case, teacher-led professional development can be a meaningful way of teachers collaboratively design, implement, and evaluate suitable SBA projects for the upcoming phase of the curriculum.

Findings on Students' Conceptions

My analysis of students' conceptions suggests that their conception of the role of appreciation of mathematics suggest their understanding of the intellectual necessity of the mathematical knowledge they learn in school. My findings suggest that students seemed to value the need of proof based on their teachers' construction of proof arguments. Students claimed that their teachers' effective application of previously proven results to prove a new result allowed them to appreciate the usefulness of their previous knowledge and the relationships between their new and previous knowledge. Students also claimed that their teachers' demonstration of proof writing along with the meta-discursive rules for proving allowed them to develop their own proof writing practices. These habitual inclinations for students to write proofs that provide insight and allow all readers to follow the author's line of reasoning is important for students' future metacognitive development of proof writing skills.

The informal GCNE opportunities in the CSEC examinations do provide opportunities for students to construct proof arguments. Although these questions do not explicitly state the word "Prove," they provide the necessary scaffolding to develop rationales for reasoning that could eventually go beyond the border of this activity and evolve into formal proof arguments. My analysis of the interviews indicates that students do consider the GCNE exercises as

opportunities for constructing a proof. Students stated that the requirement for providing supporting explanations for one's calculations is equivalent to a proof that explains the validity of a result. This finding is important because it suggest students' meta-cognitive knowledge about the role of explanations in communicating one's mathematical thinking.

Implications and Future Work on Students' Conceptions and Learning

Research on reform efforts in mathematics education recommends approaches to teaching mathematics that is greatly supported by students' ideas and intuitions (Ball, 1994; Remillard 1999). In Trinidad and Tobago, the reformers suggest that instruction should be aimed at making student thinking central (Republic of Trinidad and Tobago, Ministry of Education, 2003, 2009). These recommendations suggest a more student-centered approach to instruction. As a result, the teachers have the dilemma of being able to navigate through students' unanticipated perspectives of the content. A teacher's understanding of students' thinking can provide coherence to a teacher's pedagogical content knowledge of subject matter, curriculum, and pedagogy (Carpenter, Fennema, & Franke, 1996; Hiebert, 1997). Therefore the findings of Chapter 4 can help both teachers and policy makers understand students thinking about proof and its role in the curriculum. The findings about students' conceptions of GCNE type exercises in the CSEC examinations supports the need for future investigation of students' thinking when solving GCNE exercises. This future work can potentially provide researchers and teachers with an understanding of the cognitive complexity of these exercises and their possible role in developing students reasoning and proving skills.

Limitations of the Study

The studies comprising this dissertation had one major limitation. First, the setting and time constraints of the classroom observations facilitated my selection of a small sample of

teachers and students. Although my rationale for the selection of the school sites indicated three levels of students' ability and school environments, it is possible that there may be variations within each level. For example, two teachers at two different government elite schools may employ different pedagogical approaches that either conform with or differ from the reformers' instructional recommendations. Additionally, different teachers at the same school may use the same curriculum materials differently. Therefore it is imperative that in order to make generalizable claims about the implications of the reform on teaching, future work must include larger representative samples.

Due to the limited time for observing teachers in this study, it is possible that teachers may have exhibited their best practices during my observations. Kennedy (2005) claims that in her work with teachers, there was the possibility of teachers exhibiting their best instructional practices when being observed by researchers. Therefore it would be useful to observe teachers over a longer period of time. The longer observational study could possibly show how the teachers' practices may evolve, change, or develop over the duration of an academic year or term (i.e., semester). Kennedy (2005) and Hufferd-Ackels et al. (2004) support the idea of having teachers reflect on their teaching practices over longer periods of observations. Teachers can have an opportunity to assess and enhance their instructional practices while receiving research-based instructional support and guidance. This could lead to meaningful forms of personal professional development and instructional change during longer observational periods.

Another limitation of this work was my consideration of evidence of students' learning through their conceptions of proof. In mathematics education research inquiries into students' conceptions often occur without paying attention to how these conceptions are influenced by social, cultural, and mathematical identity. It seems beneficial to consider ideas of students'

mathematical identity when examining their conceptions of proof. Several research projects have considered how students' mathematical experiences and conceptions could be shaped by their personal identities and backgrounds (e.g., Boaler, 2002; Nasir, 2002). Therefore a more pertinent measure of students learning may include theoretical frameworks for understanding students' conceptions in ways that highlight their backgrounds, schooling experiences, and mathematical identity.

Overall Research Contribution and Trajectory

The findings of my current research projects could help researchers understand the implications of the recent reform recommendations in mathematics education on the content, students' learning, teachers' pedagogical choices and classroom practices. In my work thus far I established a research foundation to examine the following (a) the implications of reform-based curriculum on teaching and student learning of mathematics, (b) the development of pedagogical approaches for reform-based instruction of reasoning and proof, and (c) the influence of curriculum materials and teachers' expectations on students' conceptions of reasoning and proof. The aforementioned areas provide an initial lens for looking at issues of the interaction between content, teaching, and student learning. In the introduction of this dissertation I used Cohen, Raudenbush, and Ball's (2003) definition of instruction to guide my initial inquiry into the reform of mathematics. With this model, I intend to continue my research into the teaching and learning of reasoning and proof in other areas of mathematics such as Algebra. I intend to examine the nature of the opportunities afforded in Algebra at the secondary and post-secondary mathematics curriculum in Trinidad and Tobago. Over the next two years the government of Trinidad and Tobago will be engaging in the final stage of the reform of the mathematics curriculum. This will include the adoption of new school-based assessment projects in

mathematics. These projects will include the use of real-world scenarios that foster opportunities for students to use pattern identification, making mathematical hypotheses, and developing arguments to support their hypotheses about mathematical phenomenon. As a result, students will be given opportunities to develop the skills and attitudes deemed necessary for critical thinking and creative development of mathematical ideas in Geometry and Algebra. It is my intention through the integration of cultural artifacts of Trinidad and Tobago such as the steel pan, carnival costume design, and Geometry or Algebra concepts to create suitable sample projects for these school-based assessments. Through collaborative efforts with teachers and other stakeholders I intend to conduct research that provides practitioner-based support for teachers. This will include sample projects conducted in actual classrooms, measuring students' outcomes and examining the type of instructional practices surrounding the teaching of these sample school-based assessment projects. The possible outcomes of the aforementioned proposed research endeavors can help policy makers, textbook writers, teachers, and external examination councils such as the Caribbean Examination Council (CXC) design curriculum materials that achieve the objectives of the reform and enrich the mathematical experiences of students.

Concluding Remarks

The findings of this study could help researchers understand the implications of the recent reform recommendations on teaching and learning of proof in Trinidad and Tobago. As I explained earlier, reform provided the context for examining the existing instructional practices surrounding reasoning and proof. Based on my findings, teachers in this dissertation exhibited pedagogy aligned with the reformers' vision for teaching and learning of reasoning and proof. In conclusion, I posit that reform is not the cause and effect of the observed teaching and learning

but it provided a lens for observing actual classroom teaching, implementation of the national curriculum, and the possible influences of the written curriculum on students' learning. Additionally, combining analyses of the written curriculum with analyses of observations of actually teaching episodes can be useful to policy makers, education stakeholders (e.g., textbook authors and examination bodies), curriculum designers, and teachers in creating future opportunities to support students learning of proof in Trinidad and Tobago. International studies such as this dissertation which apply frameworks and results from the US to different educational situations such as Trinidad and Tobago's, examine how students in different countries are afforded opportunities to prove, and investigate the nature of the association of textbooks and teachers' decision-making and instructional practices, would help in understanding the relationship between content, teaching and the implications of this relationship for student learning. Furthermore, this study adds to the existing literature on the relationship between teaching and learning by helping to bridge the gap between policy and practice.

REFERENCES

Ayres, P., & Sweller, J. (1990). Locus of difficulty in multistage mathematics problems. *The American Journal of Psychology*, 167-193.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. *Mathematics, teachers and children*, *216*, 235.

Balacheff, N. (2010). *Bridging knowing and proving in mathematics: A didactical perspective* (pp. 115-135). Springer US.

Ball, S. (1994). *Education reform: A critical and post-structural approach*. McGraw-Hill Education (UK).

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. *A research companion to principles and standards for school mathematics*, 27-44.

Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is: Or might be: The role of curriculum materials in teacher learning and instructional reform? *Educational researcher*, 6-14.

Ball, D. L., Hoyles, C., Jahnke, H. N., & Movshovitz-Hadar, N. (2002). The teaching of proof. In *Proceedings of the International Congress of Mathematicians* (Vol. 3).

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. *Effective mathematics teaching*, *1*, 27-46.

Beaton, A. E., Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., Kelly, D. L., & Smith, T. A. (1996). Mathematics achievement in the middle school years. Boston: Centre for the Study of Testing. *Evaluation and Educational Policy, Boston College*.

Begle, E. G. (1973). Some Lessons Learned by SMSG. *Mathematics Teacher*, *66*(3), 207-14.

Bell, A. W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics, 7*, 23-40.

Bell-Hutchinson, C. (2004). *Teaching-for-thinking: The implementation of thinking-focused pedagogy in two Grade 8 mathematics classrooms in Jamaica*. Unpublished doctoral dissertation, The University of the West Indies, Mona, Jamaica.

Bieda, K. N. (2010). Enacting proof-related tasks in middle school mathematics: Challenges and opportunities. *Journal for Research in Mathematics Education*, 351-382.

Boaler, J. (2002). The development of disciplinary relationships: Knowledge, practice and identity in mathematics classrooms. *For the learning of mathematics*, *22*(1), 42-47.

Borko, H., & Putnam, R. (1998). Professional development and reform-based teaching: Introduction to the theme issue. *Teaching and Teacher Education*, *14*(1), 1-3.

Brown, L. I. (2005). Gender and academic achievement in math: An examination of the math performance data on seven to nine year olds in Trinidad and Tobago. University of the West Indies St Augustine.

Bruner, J. S. (1996). *The culture of education*. Harvard University Press.

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H. G. Steiner (Ed.), *Theory of mathematics education* (ICME-5 Topic Area and miniconference) (pp. 110–119). Bielefeld, Germany: Institute für Didaktik der Mathematik, Universität Bielefeld.

Cai, J., & Cirillo, M. (2014). What do we know about reasoning and proving? Opportunities and missing opportunities from curriculum analyses. *International Journal of Educational Research*, *64*, 132-140.

Cai, J., & Howson, G. (2013). Toward an international mathematics curriculum. In *Third International Handbook of Mathematics Education* (pp. 949-974). Springer New York.

Campbell, C. C. (1997). *Endless education: Main currents in the education system of modern Trinidad and Tobago, 1939-1986*. University of West Indies Press.

Caribbean Examination Council. (2012). Caribbean *Secondary Education Certificate Examinations Syllabus: Mathematics*. St. Michael, Barbados: Author.

Caribbean Examination Council. (2013). *Subject Award Committee Report: Mathematics, May/June.* St. Michael, Barbados: Author.

Caribbean Examination Council. (2014). *Subject Award Committee Report: Mathematics, May/June.* St. Michael, Barbados: Author.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, 3-20.

Castro, C. D. M., Carnoy, M., & Wolff, L. (2000). *Secondary schools and the transition to work in Latin America and the Caribbean*. Inter-American Development Bank.

Chandler, S., Smith, E., Ali, F. W., Layne, C., & Mothersill, A. (2008) *Mathematics for CSEC*. United Kingdom: Nelson Thornes.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational studies in mathematics*, *24*(4), 359-387.

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom*. Teachers College Press.

Chazan, D., & Houde, R. (1989). *How To Use Conjecturing and Microcomputers To Teach Geometry*. National Council of Teachers of Mathematics, 1906 Association Dr., Reston, VA 22091.

Cirillo, M. (2008). *On becoming a geometry teacher: A longitudinal case study of one teacher learning to teach proof*. ProQuest.

Cirillo, M. (2009). Ten Things to Consider when Teaching Proof. *Mathematics Teacher*, *103*(4), 250-257.

Cirillo, M., & Herbst, P. G. (2012). Moving toward More Authentic Proof Practices in Geometry. *Mathematics Educator*, *21*(2), 11-33.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives in mathematical development. *Educational Researcher, 23*, 13-20.

Cobb, P., & Bowers, J. (1999). Cognitive and situated learning perspectives in theory and practice. *Educational researcher*, *28*(2), 4-15.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational psychologist*, *31*(3-4), 175-190.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*, 2-33.

Cohen, D. K. (2011). *Teaching and its predicaments*. Harvard University Press.

Cohen D. K. & Hill, H.C. (2000) Instructional policy and classroom performance: The mathematics reform in California. *Teachers College Record* 102, 296-345.

Cohen, D. K., Raudenbush, S.W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25* (2), 119-142.

Conway, P. F. (2002). Learning in communities of practice: Rethinking teaching and learning in disadvantaged contexts. *Irish Educational Studies 21,* 3, 61-92.

Coe, R., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal*, *20*(1), 41-53.

Cox, R. L. (2004). Using Conjectures to Teach Students the Role of Proof. *Mathematics Teacher*, *97*(1), 48.

Cuban, L. (1984). *How Teachers Taught: Constancy and Change in American Classrooms, 1890-1980. Research on Teaching Monograph Series*.

Cuban, L. (1990). Reforming again, again, and again. *Educational researcher*, *19*(1), 3-13.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior*, *15*(4), 375-402.

Da Ponte, J. P., & Chapman, O. (2006). Mathematics teachers' knowledge and practices. *Handbook of research on the psychology of mathematics education*, 461-494.

Darling-Hammond, L. (2008). Teacher learning that supports student learning. *Teaching for intelligence*, *2*, 91-100.

Davis, J. D. (2012). An examination of reasoning and proof opportunities in three differently organized secondary mathematics textbook units. *Mathematics Education Research Journal*, *24*(4), 467-491.

De Lisle, J., Smith, P., & Jules, V. (2005). Which males or females are most at risk and on what? An analysis of gender differentials within the primary school system of Trinidad and Tobago. *Educational Studies*, *31*(4), 393-418.

De Villiers, M. (1999). The role and function of proof with Sketchpad. *Rethinking Proof with The Geometer's Sketchpad*. Emerville, CA: Key Curriculum Press.

De Villiers, M. (2010). *Experimentation and proof in mathematics* (pp. 205-221). Springer US.

Dickerson, D. S. (2008). *High school mathematics teachers' understandings of the purposes of mathematical proof*. Syracuse University). *ProQuest Dissertations and Theses,* 240-n/a. Retrieved from http://search.proquest.com/docview/304371864?accountid=14553. (304371864).

Ding, L., & Jones, K. (2006). Teaching geometry in lower secondary school in Shanghai, China. *Proceedings of the British Society for Research into Learning Mathematics*, *26*(1), 41-46.

Dörfler, W., & McLone, R. R. (1986). Mathematics as a school subject. In *Perspectives on mathematics education* (pp. 49-97). Springer Netherlands.

Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics*, *38*(1-3), 85-109.

Dreyfus, T., & Hadas, N. (1987). Euclid May Stay--and Even Be Taught. *Learning and teaching geometry, K-12*, 47-58.

Epp, S. S. (1998). A unified framework for proof and disproof. *The Mathematics Teacher*, 708-713.

Fernandez, C., & Yoshida, M. (2012). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. Routledge.

Fischbein, E. (1982). Intuition and proof. *For the learning of mathematics*, 9-24.

Floden, R. E. (2002). The measurement of opportunity to learn. In A. C. Porter & A. Gamoran (Eds.), *Methodological advances in cross-national surveys of educational achievement* (pp. 231–266). Washington, DC: National Academy Press.

Forman, E. A., Stein, M. K., Brown, C. A., & Larreamendy-Joerns, J. (1995, April). *The socialization of mathematical thinking: The role of institutional, interpersonal, and*

discursive contexts. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Franke, M. L., & Kazemi, E. (2001). Learning to teach mathematics: Focus on student thinking. *Theory into practice*, *40*(2), 102-109.

Fernandez, C., & Yoshida, M. (2012). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. Routledge.

Fujita, T., & Jones, K. (2014). Reasoning-and-proving in geometry in school mathematics textbooks in Japan. *International Journal of Educational Research*, *64*, 81-91.

George, J., Mohammed, J., & Quamina-Aiyejina, L. (2003). Teacher identity in an era of educational reform: The case of Trinidad and Tobago. *Compare*, *33*(2), 191–206.

González, G., & Herbst, P. G. (2006). Competing arguments for the geometry course: Why were American high school students supposed to study geometry in the twentieth century? *International Journal for the History of Mathematics Education*, *1*(1).

González, G., & Herbst, P. G. (2009). Students' conceptions of congruency through the use of dynamic geometry software. *International Journal of Computers for Mathematical Learning*, *14*(2), 153-182.

Greeno, J. G., & Goldman, S. V. (Eds.). (2013). *Thinking practices in mathematics and science learning*. Routledge.

Greeno, J. G., & MMAP. (1997). Theories and practice of thinking and learning to think. *American Journal of Education, 106*, 85-126.

Greer, A., & Layne, C. (1994). *Certificate Mathematics*. England: Stanley Thornes.

Gross, N., Giaquinta, J., & Bernstein, M. (1971). Implementing educational innovations. *New York: Basic*.

Grouws, D. A., Smith, M. S., & Sztajn, P. (2004). The preparation and teaching practices of United States mathematics teachers: Grade 4 and 8. In P. Kloosterman & F. K. Lester, Jr. (Eds.), *Results and interpretations of the 1990–2000 mathematics assessments of the National Assessment of Educational Progress* (pp. 221–267). Reston, VA: National Council of Teachers of Mathematics.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, *21*(1), 6-13.

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of mathematics*, 42-49.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational studies in mathematics*, *44*(1-2), 5-23.

Hanna, G., & de Bruyn, Y. (1999). Opportunity to learn proof in Ontario grade twelve Mathematics texts. *Ontario Mathematics Gazette*, *37*(4), 23-29.

Hanna, G., & de Villiers, M. (2008). ICMI Study 19: Proof and proving in mathematics education. *ZDM*, *40*(2), 329-336.

Hanna, G., & Jahnke, H. N. (1996). Proof and proving. In *International handbook of mathematics education* (pp. 877-908). Springer Netherlands.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *Research in collegiate mathematics education III*, *7*, 234-282.

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, 38-42.

Healy, L., & Hoyles, C. (1998). *Justifying and proving in school mathematics. Summary of the results from a survey of the proof conceptions of students in the UK*. Research Report Mathematical Sciences, Institute of Education, University of London.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for research in mathematics education*, 396-428.

Heinze, A., Cheng, Y. H., & Yang, K. L. (2004). Students' performance in reasoning and proof in Taiwan and Germany: Results, paradoxes and open questions. *ZDM*.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 524-549.

Herbst, P. G. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education*, 176-203.

Herbst, P., & Brach, C. (2006). Proving and doing proofs in high school geometry classes: What is it that is going on for students? *Cognition and Instruction*, *24*(1), 73-122.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, *24*(4), 389-399.

Hersh, R. (2009). What I would like my students to already know about proof. In D.A., Stylianou, D. A., M.L., Blanton, & E. J. Knuth (Eds.), *Teaching and Learning Proof Across the Grades K-16 Perspective*. NY: Routledge.

Hiebert, J. (1997). *Making sense: Teaching and learning mathematics with understanding*. Heinemann, 361 Hanover Street, Portsmouth, NH 03801-3912.

Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one?. *Educational researcher*, *31*(5), 3-15.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second handbook of research on mathematics teaching and learning*, *1*, 371-404.

Hiebert, J. & Stigler, J. W. (2004). A world of difference: Classrooms abroad provide lessons in teaching math and science. *Journal of Staff Development, 10-15.*

Hoyles, C. (1997). The curricular shaping of students' approaches to proof. *For the learning of mathematics*, *17*(1), 7-16.

Hsu, H. Y., & Silver, E. A. (2014). Cognitive complexity of mathematics instructional tasks in a Taiwanese classroom: An examination of task sources. *Journal for Research in Mathematics Education*, *45*(4), 460-496.

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for research in mathematics education*, 81-116.

Hurley, P. J. (2006). *A concise introduction to logic* (9th ed.). Belmont, CA: Thomson Learning.

Husen, T. (1967). *International Study of Achievement in Mathematics: A comparison of twelve countries* (Vol. 1). New York, NY: John Wiley & Sons.

Jackson, C. K. (2010). Do students benefit from attending better schools? Evidence from rule‐based student assignments in Trinidad and Tobago. *The Economic Journal*, *120*(549), 1399-1429.

Jackson, C. K. (2012). Single-sex schools, student achievement, and course selection: Evidence from rule-based student assignments in Trinidad and Tobago. *Journal of Public Economics*, *96*(1), 173-187.

Jennings, Z. (1993). Curriculum change in school systems in the Commonwealth Caribbean: Some implications for the management of curriculum development. *International Journal of Educational Development*, *13*(2), 131–143.

Jones, K., Fujita, T., & Ding, L. (2006). Informing the pedagogy for geometry: learning from teaching approaches in China and Japan. *Proceedings of the British Society for research into Learning Mathematics*, *26*(2), 109-114.

Kennedy, M. M. (2005). *Inside teaching: How classroom life undermines reform*. Harvard University Press.

Kieran, C. (1992). The learning and teaching of algebra. *Handbook of research on mathematics teaching and learning*, 390-419.

Knipping, C. (2008). A method for revealing structures of argumentations in classroom proving processes. *ZDM*, *40*(3), 427-441.

Knuth, E. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, *5*, 61-88.

Küchemann, D., & Hoyles, C. (2009). From computational to structural reasoning: tracking changes over time. In D.A., Stylianou, D. A., M.L., Blanton, & E. J. Knuth (Eds.), *Teaching and Learning Proof Across the Grades K-16 Perspective*. NY: Routledge.

Kunimune, S., Fujita, T., & Jones, K. (2009). " Why do we have to prove this?" Fostering students' understanding of 'Proof" in geometry in Lower Secondary School. *Mathematics Education*, 256.

Lakatos, I. (1976). Proofs and refutations: The logic of mathematical discovery (edited by J. Worrall & E. Zahar).

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American educational research journal*, *27*(1), 29-63.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Philips, E. D. (1998/2004). *Connected Mathematics Project*. Menlo Park, CA: Dale Seymour Publications.

Lave, J., & Wenger, W. (1991). *Situated learning: Legitimate peripheral participation.* New York: Cambridge University Press.

Lewin, K., Keller, C., & Taylor, E. (2002). *Teacher Education in Trinidad and Tobago: Costs Financing and Future Policy*. University of Sussex Institute of Education, Centre for International Education.

Lewis, C. C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Research for Better Schools.

Li, Y. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education*, 234-241.

London, N. A. (1989). Selecting students for secondary education in a developing society: The case of Trinidad and Tobago. *McGill Journal of Education/Revue des sciences de l'éducation de McGill*, *24*(003).

London, N. A. (1997). Socio-politics in effective curriculum change in a less developed country: Trinidad and Tobago. *Curriculum Inquiry*, *27*(1), 63– 80.

Lubienski, S. (2011). Mathematics education and reform in Ireland: An outsider's analysis of Project Maths. *Bulletin of the Irish Mathematical Society*, *67*, 27-55.

Lucast, E. K. (2003) Proof as method: a case for proof in mathematics curricula. (MS thesis) (Ann Arbor, USA: ProQuest information and Learning).

Magone, M. E., Cai, J., Silver, E. A., & Wang, N. (1994). Validating the cognitive complexity and content quality of a mathematics performance assessment. *International Journal of Educational Research*, *21*(3), 317-340.

Maher, C. (2009). Children's reasoning: Discovering the idea of mathematical proof. In M. Blanton, Stylianou, D., & Knuth, E. (Eds.) *Teaching and learning mathematics proof across the grades: A K-16 perspective* (pp. 120-132). NY: Routledge.

Marrades, R., & Gutiérrez, Á. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational studies in mathematics*, *44*(1-2), 87-125.

Mariotti, M. A. (2006). Proof and proving in mathematics education. *Handbook of research on the psychology of mathematics education: Past, present and future*, 173-204.

Martin, T. S., & McCrone, S. S. (2003). Classroom factors related to geometric proof construction ability. *Mathematics Educator, 7* (1), 18-31.

Martin, T. S., McCrone, S. M. S., Bower, M. L. W., & Dindyal, J. (2005). The interplay of teacher and student actions in the teaching and learning of geometric proof. *Educational Studies in Mathematics*, *60*(1), 95-124.

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, 41-51.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*(3), 277-289.

McCrone, S. M., & Martin, T. S. (2004). Assessing high school students' understanding of geometric proof. *Canadian Journal of Math, Science & Technology Education*, *4*(2), 223-242.

McCrone, S. M., & Martin, T. S. (2009). Formal proof in high school geometry: Student perceptions of structure, validity, and purpose. In D.A., Stylianou, D. A., M.L., Blanton, & E. J. Knuth (Eds.), *Teaching and Learning Proof Across the Grades K-16 Perspective*. NY: Routledge.

Moyer, J. C., Cai, J., Wang, N., & Nie, B. (2011). Impact of curriculum reform: Evidence of change in classroom practice in the United States. *International Journal of Educational Research*, *50*(2), 87-99.

Muis, K. R. (2004). Personal epistemology and mathematics: A critical review and synthesis of research. *Review of Educational Research*, *74*(3), 317-377.

Nasir, N. I. S. (2002). Identity, goals, and learning: Mathematics in cultural practice. *Mathematical thinking and learning*, *4*(2-3), 213-247.

National Council of Teachers of Mathematics. (NCTM). (2000). Principles and Standards for School Mathematics. Reston, Va.: Author.

Noddings, N. (1990). Constructivism in mathematics education. *Journal for Research in Mathematics Education. Monograph*, Vol. 4, 7-18+195-210.

Nordström, K., & Löfwall, C. (2005, February). Proof in Swedish upper secondary school mathematics textbooks—the issue of transparency. In *Proceedings of the 4th congress of the European Society for Research in Mathematics Education (CERME4)* (pp. 448-457).

O'Connor, M. C. (1998). Can we trace the" efficacy of social constructivism"? *Review of research in education*, 25-71.

Otten, S., Gilbertson, N. J., Males, L. M., & Clark, D. L. (2014). The Mathematical Nature of Reasoning-and-Proving Opportunities in Geometry Textbooks. *Mathematical Thinking and Learning*, *16*(1), 51-79.

Polya, G. (1954). *Mathematics and plausible reasoning: vol 1: Induction and analogy in mathematics*. Oxford University Press.

Pólya, G. (1957). *How to solve it: A New Aspect of Mathematical Method. 2d Ed.* Princeton, NJ: Princeton University Press.

Polya, G. (1968). *Mathematics and Plausible Reasoning: Patterns of plausible inference* (Vol. 2). Princeton University Press.

Porter, A. C., Floden, R., Freeman, D., Schmidt, W., & Schwille, J. (1988). Content determinants in elementary school mathematics. *Perspectives on research on effective mathematics teaching*, *1*, 96-113.

Prawat, R. S. (1992). Teachers' beliefs about teaching and learning: A constructivist perspective. *American journal of education*, 354-395.

Rav, Y. (1999). Why do we prove theorems? *Philosophia mathematica*, *7*(1), 5-41.

Reid, D. (2001). Proof, Proofs, Proving and Probing: Research Related to Proof. Paper based on a Short Oral Presentation at the 25th PME Conference, Utrecht, Netherlands. http://ace.acadiau.ca/~dreid/publications/proof/proof.htm

Reid, D. (2005). The meaning of proof in mathematics education. Paper presented at CERME4.

Remillard, J. T. (1999). Curriculum materials in mathematics education reform: A framework for examining teachers' curriculum development. *Curriculum Inquiry*, *29*(3), 315-342.

Remillard, J. T. (2000). Can curriculum materials support teachers' learning? Two fourth-grade teachers' use of a new mathematics text. *The Elementary School Journal*, 331-350.

Republic of Trinidad and Tobago. Ministry of Education. Secondary Education Modernization Programme. (2003). *Secondary school curriculum: Form three mathematics*. Port of Spain: Author.

Republic of Trinidad and Tobago. Ministry of Education. Secondary Education Modernization Programme. (2009). *Secondary school curriculum: Form Four and Five mathematics*. Port of Spain: Author.

Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics*, *83*(3), 323-340.

Schmidt, W. H., McKnight, C. C., Houang, R. T., Wang, H., Wiley, D. E., Cogan, L. S., & Wolfe, R. G. (2001). *Why Schools Matter: A Cross-National Comparison of Curriculum and Learning. The Jossey-Bass Education Series*. Jossey-Bass, 989 Market Street, San Francisco, CA 94103-1741.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. *Handbook of research on mathematics teaching and learning*, 334-370.

Schoenfeld, A. H. (1994). What do we know about mathematics curricula? *The Journal of Mathematical Behavior*, *13*(1), 55-80.

Schorr, R. Y., Firestone, W. A., & Monfils, L. (2003). State testing and mathematics teaching in New Jersey: The effects of a test without other supports. *Journal for Research in Mathematics Education*, 373-405.

Schommer, M. (1994). Synthesizing epistemological belief research: Tentative understandings and provocative confusions. *Educational psychology review*, *6*(4), 293-319.

Schwille, J., Porter, A., Floden, R., Freeman, D., Knapp, L., Kuhs, T., & Schmidt, W. (1983). Teachers as policy brokers in the content of elementary school mathematics. In L. Shulman and G. Sykes (Eds.), *Handbook of teaching and policy* (pp. 370-391). New York: Longman.

Sears, R., & Chávez, Ó. (2014). Opportunities to engage with proof: the nature of proof tasks in two geometry textbooks and its influence on enacted lessons. *ZDM*, *46*(5), 767-780.

Senk, S. L. (1985). How well do students write geometry proofs? *The mathematics teacher*, 448-456.

Senk, S. L., & Thompson, D. R. (Eds.). (2003). *Standards-based school mathematics curricula: What are they? What do students learn?*. Mahwah, NJ: Lawrence Erlbaum Associates.

Senk, S. L., Thompson, D. R., & Johnson, G. (2008, July). Reasoning and proof in high school textbooks from the USA. In *11th International Congress on Mathematics Education, Monterey, Mexico*.

Sfard, A. (2000). On reform movement and the limits of mathematical discourse. *Mathematical thinking and learning*, *2*(3), 157-189.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for research in mathematics education*, 114-145.

Sowder, L., & Harel, G. (1998). Types of Students' Justifications. *Mathematics Teacher*, *91*(8), 670-75.

Stacey, K., & Vincent, J. (2009). Modes of reasoning in explanations in Australian eighth-grade mathematics textbooks. *Educational Studies in Mathematics*, *72*(3), 271-288.

Stake, R. E. (1978). Case Studies in Science Education, Volume I: The Case Reports.

Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics

classroom. *Cognition and Instruction*, *25*, 161-217.

Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifacted role in middle grades mathematics classrooms. *The Journal of Mathematical Behavior*, *31*(4), 447-462.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American educational research journal*, *33*(2), 455-488.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In Lester, F. K. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1). Charlotte, NC: IAP.

Stigler, J., & Hiebert, J. (1997). Understanding and Improving Classroom Mathematics Instruction: An Overview of the TIMSS Video Study. *Insights from TIMSS*, 52.

Stodolsky, S. S. (1988). *The subject matters: Classroom activity in math and social studies*. University of Chicago Press.

Stylianides, G. J. (2005). *Investigating students' opportunities to develop proficiency in reasoning and proving: A curricular perspective*. University of Michigan.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, 289-321.

Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical thinking and learning*, *11*(4), 258-288.

Taylor, J., & Cox, B. D. (1997). Microgenetic Analysis of Group-Based Solution of Complex Two-Step Mathematical World Problems by Fourth Graders. *The Journal of the Learning Sciences*, *6*(2), 183-226.

Thompson, D. R., & Senk, S. L. (2014). The same geometry textbook does not mean the same classroom enactment. *ZDM*, *46*(5), 781-795.

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education*, *43*, 253–295.

Thurston, W. (1995). On proof and progress in mathematics. *For the Learning of Mathematics*, 15(1), 29–37.

Tinto, P. (1988, April). Students' views on learning proof in high school geometry. In *meeting of the American Educational Research Association, New Orleans*.

Toolsie, R. (2009) *Mathematics a complete course.* Trinidad, West Indies: Caribbean Educational.

Trends in International Mathematics and Science Study (TIMSS). (1998). *Data almanacs for achievement items: Percent correct on the advanced mathematics items: Final year of secondary school*. Retrieved from http://timss.bc.edu/timss1995i/TIMSSPDF/pc_alm95/m12alm95.pdf.

Trinidad and Tobago (1994). *National Task Force on Education. Education policy paper (1993-2003)* (White paper). Port of Spain, Trinidad.

Trinidad and Tobago. Ministry of Education (2005). *Quest for excellence: Quality standards for education in Trinidad and Tobago*: *A Ministry of Education Green Paper — first revision*. Port of Spain, Trinidad.

Trinidad and Tobago. Ministry of Education (2006). *Curriculum policy: pre-primary to secondary education*; draft. Port of Spain, Trinidad, 2006

Trinidad and Tobago. Ministry of Education (2007). *The national model for education in Trinidad and Tobago (Early childhood, primary and secondary)*; draft. Port of Spain,

Trinidad.

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Voigt, J. (1985). Patterns and routines in classroom interaction. *Researches en Didactique de Mathématiques*, *6*(1), 69–118.

Vygotsky, L. S. (1978). Mind and society: The development of higher mental processes. Cambridge MA: Harvard University Press.

Watanabe, T. (2002). Learning from Japanese Lesson Study. *Educational leadership*, *59*(6), 36-39.

Weber, K., Maher, C., Powell, A., & Lee, H. S. (2008). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics*, *68*(3), 247-261.

Wilson, S. M. (2003). *California dreaming: Reforming mathematics education*. Yale University Press.

Wolff, L., & Castro, C. D. M. (2000). *Secondary education in Latin America and the Caribbean: The challenge of growth and reform*. Inter-American Development Bank.

Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for research in mathematics education*, 171-191.

Wood, T., Cobb, P., & Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal*, *28*(3), 587-616.

Woodward, J. (2004). Mathematics Education in the United States Past to Present. *Journal of Learning Disabilities*, *37*(1), 16-31.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentations and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*, 458-477.

Yackel, E., & Hanna, G. (2003). Reasoning and proof. *A research companion to principles and standards for school mathematics*, 227-236.

APPENDIX A:

CLASSROOM OBSERVATION PROTOCOL

Measure 2: Observation Protocol for IRB— The Classroom Nature of The Teaching of Reasoning and Proof in Trinidad and Tobago

Date:
Time of class observation:
Time of observation protocol being filled out:
Number of students present:
Teacher Pseudonym_____

In this study I intend to observe the classroom norms teachers establish during the teaching of reasoning and proof in Geometry. I am also interested in teachers' use of curriculum material

Overall Description of the Classroom

1. What topic did the teacher cover today?

2. What chapter/section of the textbook did the lesson cover?

Use of Curricula Materials

1. How and when does the teacher use the textbook during the lesson?

2. How and when does the teacher use CSEC mathematics examination preparatory materials during the lesson?

3. What type of problems does the teacher assign for in class practice?

4. How did the teacher assess students' progress on any problems assigned during the lesson?

Classroom Micro-culture

1. What classroom mathematical practices are established during the lesson?

2. What sociomathematical norms are established during the lesson?

3. How and when are the sociomathematical norms established during the lesson?

4. What elements of reform-oriented instruction are evident during the lesson (This includes the type of questions posed, explanation of mathematical thinking, sources of mathematical ideas, responsibility for learning)?

Reasoning and Proving

1. What type of reasoning activities took place during the lesson? (These include pattern identification, making conjectures, constructing proof-type and non-proof type arguments).

NOTES:

APPENDIX B:

PRE-OBSERVATION INTERVIEW PROTOCOL

Measure 1: Pre-Lesson Interview Protocol for Teachers before the lesson
Interview Protocol for IRB— The Classroom Nature of The Teaching of Reasoning and Proof in Trinidad and Tobago
Pseudonym: _____
General Protocol: Semi-structured interview
Interviewer: Thank you for agreeing to participate in this study. By participating in this study, you may have an opportunity to reflect on your own teaching experiences. Participation is completely voluntary and your choice to participate or not will not impact your employment or status at the Ministry of Education. I do not anticipate any risk greater than normal life. You can decide to withdraw from the study any time. This is not a test; there is no right or wrong answer. The main focus is your opinion. I do hope you enjoy this experience and thank you for your time.

Questions:

1. Can you tell me something about the class you are teaching today? How are the students?

2. What topic do you plan to teach today?

3. What are your goals for this lesson?

4. What strategies are you implementing to achieve these goals?

5. Can you tell me about any activities that engage your students in doing reasoning and proofs?

6. Which textbook do you prefer to use when teaching? Can you tell me more about why you prefer it to the other available options?

7. Did you use other resources or materials in preparing this lesson? (If yes, can you describe the materials and how you used them)?

APPENDIX C

POST-OBSERVATION INTERVIEW PROTOCOL

Measure 1: Post-Lesson Interview Protocol for Teachers before the lesson
Interview Protocol for IRB— The Classroom Nature of the Teaching of Reasoning and Proof in Trinidad and Tobago.
Pseudonym: _____
General Protocol: Semi-structured interview
Interviewer: Thank you for agreeing to participate in this study. By participating in this study, you may have an opportunity to reflect on your own teaching experiences. Participation is completely voluntary and your choice to participate or not will not impact your employment or status at the Ministry of Education. I do not anticipate any risk greater than normal life. You can decide to withdraw from the study any time. This is not a test; there is no right or wrong answer. The main focus is your opinion. I do hope you enjoy this experience and thank you for your time.

Questions:

1. What do you think went well in the lesson today?

2. Is there anything that you would do differently?

3. How are you planning to assess students' understanding of the concepts in this lesson?

4. Are there any problems or challenges you've had to overcome while teaching this class? How did you resolve or deal with them?

5. In your opinion, what are the most effective ways that teachers can positively influence their students' reasoning and proving skills in mathematics?

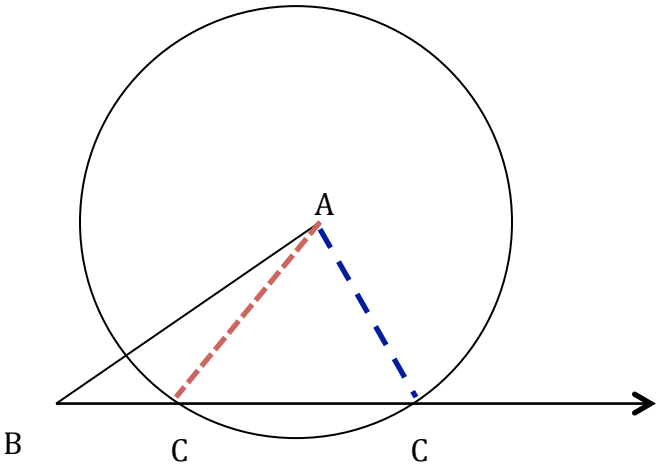
APPENDIX D:

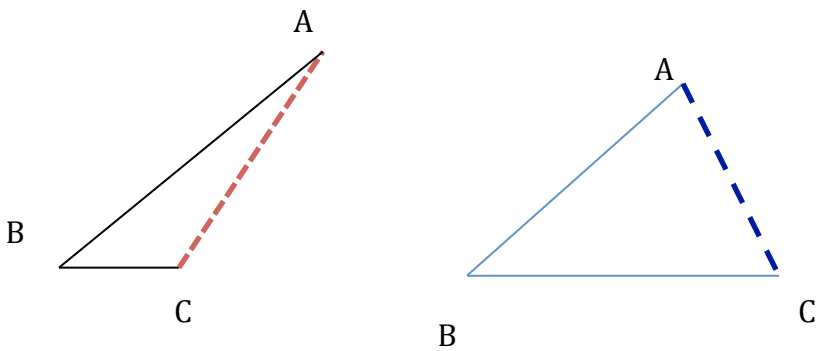
PROOF OF WHY ASS OR SSA DOES NOT PROVE CONGRUENCE

Angle-Side-Side (ASS) and Side-Side-Angle (SSA) cannot prove congruence. Both theorems represent the case where you have in a pair of triangles, two known sides and one congruent angle that is not the angle between two known sides.

Proof

First draw a ray from a point B and let us construct the known angle at B using the known side BA. Then let us construct angle B. Then using the other known length of AC as the radius of a circle you will construct, whose center is the point A. You will obtain the diagram below. Then using the points where the line intersects the ray from B you connect those two points of intersection. You will notice that they are both of the same lengths of AC because they are radii of the circle. You will notice that by the two dashed lines we have two triangles ABC, where we have a corresponding angle B and corresponding sides AC and BC . Thus if we construct the two triangles separately we have two triangles that are not congruent. Therefore we have a case where ASS or SSA does not hold.





APPENDIX E

SIX PRINICPLES OF PROOF UNDERSTANDING THEORETICAL FRAMEWORK

Theoretical framework for study 2

Six Principles of Proof Understanding (Dreyfus & Hadas, 1987; McCrone & Martin 2004)

Principle	Statement and Description of subcomponents
Principle 1 : Counterexamples and Validity	**A theorem has no exceptions** a. Knowing that a general statement is true implies true for all specific instances. b. Showing that a general statement is not true in one specific case(i.e. finding a counterexample)proves that the statement is false.
Principle 2: Roles of proof	**Two roles of proof are to convince and to explain** a. One purpose of proof is to establish the validity of a statement. b. Another purpose of proof is to provide insight into why the statement is true.
Principle 3: Generality of a proof	**A proof must be general** a. One method of proving a general statement is to show that the statement is true for every conceivable instance. b. If a proof is not exhaustive, then a direct proof, indirect proof, or visual proof must be constructed so that all statements or diagrams refer to all classes of figures that satisfy the given conditions c. Empirical evidence of one or more cases does not constitute a proof d. Checking a few specific critical cases does not constitute a proof.
Principle 4: Validity of proof and logical structure	The validity of a proof depends on its internal logic a. Hypotheses are distinct from conclusions b. The statements in a proof must be ordered to conform to the laws of reasoning to provide a path from the hypothesis to the conclusion. c. Inherent logic determines the validity of a proof, not ritualistic aspects of the

	form in which it is presented.
Principle 5: Logical equivalence	**Logical equivalence** Statements are logically equivalent to their contrapositive, but not necessarily to their converses or inverses.
Principle 6: Use of diagrams in proving	**Diagrams that illustrate statements have benefits and limitations** a. Diagrams that illustrate a statement are visual aids. One must keep in mind those aspects of the diagram that reflect the conditions of the statements and those that may vary b. Focusing on components of the diagrams may be essential for identifying the fundamental relationships that connect the hypothesis to the conclusion

APPENDIX F

STUDENT INTERVIEW PROTOCOL

Measure 1: Interview Protocol for Students
Interview Protocol for IRB— Secondary School Students' Conceptions about Proof in Trinidad and Tobago
Pseudonym: _____

General Protocol: Semi-structured interview

Interviewer: Thank you for agreeing to participate in this study. By participating in this study, you may have an opportunity to reflect on your own ideas about proof in your secondary school mathematics experiences. Participation is completely voluntary and your choice to participate or not will not impact your status or grades at school. I do not anticipate any risk greater than normal life. You can decide to withdraw from the study any time. This is not a test; there is no right or wrong answer. The main focus is your opinion. I do hope you enjoy this experience and thank you for your time.

Questions:

SECTION 1: Roles of Proof in School Mathematics

1. Which form (grade) are you currently in?

2. What is your future plan for your studies with mathematics? Do you plan to enroll in mathematics classes in Form six (grades 11 and 12) or college?

3. In your opinion, why do you think mathematicians write proofs?

4. What would you look for to determine if a proof is correct? Or, what do you look for in a proof to be convinced by the argument?

5. In your opinion, could you provide reasons why you are taught proof in mathematics?

6. In which mathematics subject areas, have you studied proofs? If possible, please show me a proof you remember from your mathematics class?

SECTION 2: Logical Structure and Validity of Proof

Interviewer: Now I am going to show some mathematical quotes from textbooks for you to tell me what you think about them. I will ask you a question about each quote.

7. Consider the following argument:

> If two segments have the same length, then they are congruent.
> A triangle is isosceles if and only if it has at least two congruent sides.

All equilateral triangles have two segments that are congruent.

CONCLUSION: **All equilateral triangles are isosceles.**

Is the conclusion correct, based on the argument? Why or why not?

8. Consider the following statements:

Statement 1: There exist one equilateral triangle that is not isosceles.
Statement 2: There exists one isosceles triangle that is not equilateral.

Which of these statements, if true, would make the conclusion false? Why would this statement cause the conclusion to be false?

Interviewer: For the next two questions, I will show you examples of student-written arguments for proving a theorem from your textbook. I would like to hear your thoughts about them.

SECTION 3: Generality of Proof

Consider the following arguments for proving the theorem in Example 1. The theorem is taken from *Mathematics a complete course* (Toolsie, 2009)

9. Which argument would be closest to what you would write? Please explain.

10. Which argument do you suppose would receive the best mark from your teacher? Please explain.

Interviewer: Now for the last few questions, we are going to talk about proof in your textbooks and the CSEC examination.

SECTION 4: Opportunities to Learn Proof in Curricula Materials

11. Which textbook do you currently use for your form 4 and form 5 classes? Can you tell me more about the text and about the proofs they present to you?

12. Does the textbook offer questions where you can practice proofs? Can you show me a few examples if possible?

13. Does your teacher usually assign proofs from the textbook? If yes could you show me a few examples if possible?

14. Consider the question in Example 2. This question is taken from the January 2014, CSEC Mathematics examination. In your opinion why or why not would you consider an answer to this question as a proof?

15. Could you describe any support you receive besides support from your classroom teacher, to help you prepare for CSEC examination?

16. In your opinion, do you think that you are sufficiently prepared for higher-level mathematics involving proof?

Example 1

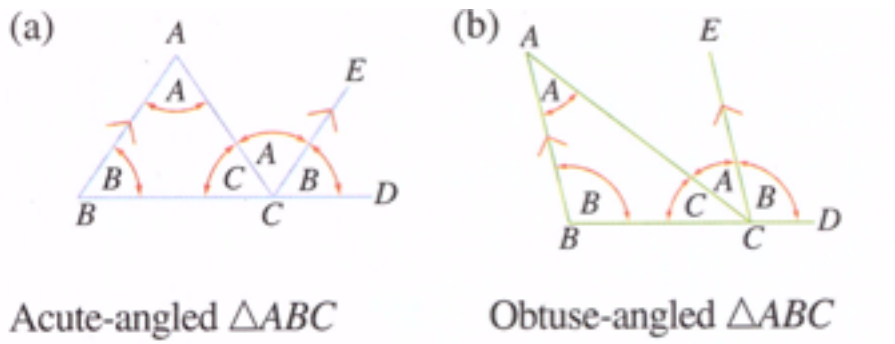
Theorem

If any side of a triangle is extended then the exterior angle so formed is equal to the sum of the two remote interior angle measures (adapted from page 445 Toolsie, R. (2009) *Mathematics a complete course).*

Argument 1

If I take a ruler and a pencil and draw any triangle say ΔABC shown below and extend the side BC to a point D. Now take my protractor and measure the exterior angle say BCD and then measure the interior angles A and B then you will observe that if you find the sum of the measures of the two angles A and B then that sum is equal to the measure of angle BCD. Once I show this for a few triangles then the theorem is true.

Argument 2

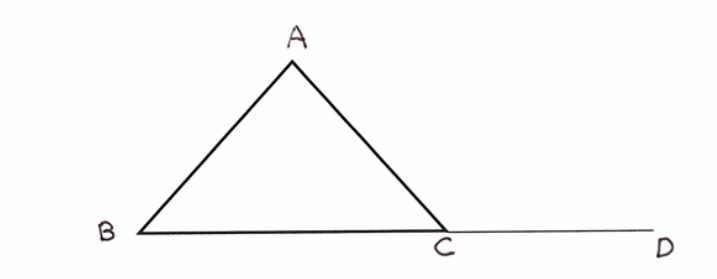


R. T. P. (Required to Prove) *that if any side of a triangle is extended then the measure of the exterior angle so formed is equal to the sum of the measures of the two opposite interior angles.*

To prove this we will need to consider all possible cases of triangles: acute-angled triangles and obtuse-angled triangle.

Considering any acute-angled triangle ABC and any obtuse-angled triangle ABC. Let us extend the side BC to D. Then draw CE parallel to BA. Now $A\hat{C}E = B\hat{A}C = \hat{A}$
because they are alternating angles resulting from the traversal AC cutting the two parallel lines BA and CE. And $E\hat{C}D = A\hat{B}C = \hat{B}$
because they are corresponding angles resulting from the traversal BC cutting the parallel lines BA and CE. Therefore
$A\hat{C}B + A\hat{C}E + E\hat{C}D = \hat{C} + \hat{A} + \hat{B} = 180^0$ (angles on a straight line)
And $A\hat{C}B = \hat{C}$
$A\hat{C}E + E\hat{C}D = \hat{A} + \hat{B}$ Subtracting $\hat{C}$ from both sides we have
$A\hat{C}D = A\hat{C}E + E\hat{C}D = B\hat{A}C + A\hat{B}C = \hat{A} + \hat{B}$. Hence the theorem is proven. Q.E.D

Argument 3

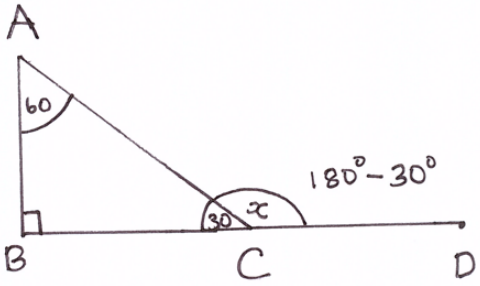


Let ABC be any triangle, and extend BC beyond C to D, so that ACD is an exterior angle.

$B\hat{A}C + A\hat{B}C + A\hat{C}B = 180^0$ [Sum of angle measures in a triangle]
$A\hat{C}D + A\hat{C}B = 180^0$ [Sum of angle measures in a linear pair]
$A\hat{C}B = 180^0 - A\hat{C}D$ [Subtraction property of equality]
$B\hat{A}C + A\hat{B}C + 180^0 - A\hat{C}D = 180^0$ [Substitution property of equality]
$B\hat{A}C + A\hat{B}C - A\hat{C}D = 0$ [Subtraction property of equality]
$B\hat{A}C + A\hat{B}C = A\hat{C}D$ [Addition property of equality]

Hence the result.

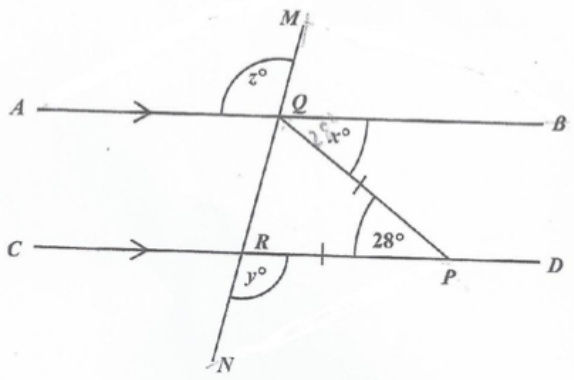
Argument 4



Let us consider any 30-60-90 degree right-angled triangle where $\hat{A} = 60^0$, $\hat{B} = 90^0$
And $\hat{C} = 30^0$
Now let us extend BC to D. Let *x* represent the exterior angle ACD. Angles C and *x* are on a straight line so
$\hat{C} + A\hat{C}D = 180^0$ [Sum of angles on a straight line]
$\hat{C} + x = 180^0$
$30^0 + x = 180^0$ [so we have an equation where we use algebra and solve for x]
$x = 180^0 - 30^0$
$x = 150^0$
And now $60^0 + 90^0 = 150^0 = x$ the exterior angle. Hence the result is proven.

Example 2

6. (a) In the figure below, **not drawn to scale,** the lines AQB and $CRPD$ are parallel and $MQRN$ is a transversal. $PQ = PR$ and angle $QPR = 28°$.



Calculate, giving reasons for your answer, the value of

(i) x **(2 marks)**

(ii) y **(2 marks)**

(iii) z. **(2 marks)**